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TWO-PHASE DISTRIBUTION IN POROUS MEDIA: AN APPLICATION OF PERCOLATION THEORY

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Abstract—The displacement of one *incompressible* fluid by *another* in a porous medium is studied using the network representation. The Bernoulli percolation model for an infinite lattice graph is utilized in the interpretation of the capillary behavior of the medium. The calculated capillary pressure–saturation relationship using the Bethe lattice results agree qualitatively with experimental data. It is shown that in some special cases of two-dimensional media no nonwetting phase removal is possible.

INTRODUCTION

Capillary displacement and the resulting distribution of fluid phases within the voids of porous media can be related to many applications such as the recovery of petroleum and the drying of porous solids. The study of capillary behavior, however, is often limited by the complexity of the pore structure and therefore requires an idealistic representation of the medium. Although a complete microscopic description would specify the manner in which the voids are imbedded and connected, along with their geometry such as size and shape, or equivalently the equation of the surface bounding the void space, an adequate representation of the pores can be obtained by using suitable approximations.

A vast majority of porous media consist of a well-interconnected three-dimensional network of pores or capillaries (Dullien 1979). Experiments indicate that the dead-end pore volume is about 1% for most consolidated media such as sandstones or sintered glass (Russell *et al.* 1947; Everett *et al.* 1950; Mysels & Stigter 1953). Thus, for macroscopically uniform, homogeneous and isotropic media (Greenkorn & Kessler 1970), henceforth referred to as *ideal*, an approximate representation is obtained by choosing an appropriate network with the medium's characteristic coordination number. The representation is made complete by assigning throat and body size distributions to the medium with suitable pore geometry for the nodes and branches (Mohanty 1981).

The representation of the microscopic structure and the prescription of pore level laws for phase occupancy make it possible to determine two-phase distribution during the immiscible displacement of one fluid by another. Microscopically, the term two-phase distribution would imply the evaluation of the class of pores occupied by one of the fluids, with respect to its assigned sizes and position in the network. Usually it is the macroscopic behavior of such capillary displacement that is of interest and is given in terms of the familiar capillary pressure–saturation relationship obtained under quasistatic conditions (Morrow 1970). Conversely, since photomicrographic information is often unavailable and is difficult to obtain, a theoretical investigation of such capillary behavior along with suitable experimental data provides the necessary insight into the pore structure.

The importance of predicting the capillary behavior, that is, the capillary pressure–saturation relationship, in terms of network modeling, motivated Fatt's (1956) study using equivalent resistor networks. Capillary behavior, given such an ideal morphological representation for random media, has been investigated using Monte Carlo simulation for injection (Chatzis & Dullien 1977) and for both injection and withdrawal of the nonwetting phase (Androutsopoulos & Mann 1979; Mann *et al.* 1981; Lin & Slattery 1982; Mohanty & Salter 1982). Capillary pressure–saturation curves have also been studied using statistical approaches such as the percolation theory based models (Larson & Morrow 1981; Heiba *et al.* (1983). Larson & Morrow (1981) considered a two-phase system with one of the phases infinitely compressible. For this case, percolation theory can be applied directly, and within the limitation of the assumptions of the network model, an exact solution for the capillary behavior of the medium can be obtained. Heiba *et al.* (1983) assumed that the fraction of the pore space invaded by the displacing phase is given by the infinite cluster density obtained from percolation theory.

Briefly, classical percolation *per se* would consider that the invading phase suddenly occupies pores for which the capillary pressure criterion (see below) is satisfied. In reality, however, the phase replacement process demands that the invading phase be connected by a path from the pore under consideration to some boundary of the medium. This was recognized by Heiba *et al.* (1983) who assigned only the infinite cluster part of the pores that satisfy the capillary pressure criterion to the invading phase. But such an analysis presumes that in a quasistatic displacement, an incompressible fluid, even after being isolated, would be removed from a pore that it occupies when the capillary pressure occupancy criterion is satisfied. Thus, in essence, without regard to the direction of saturation changes, the displaced phase is implied to be *infinitely compressible* in their analysis. Mercury invading an evacuated medium would be a typical example of such a process. Many practical situations that need to be studied, however, involve the removal of one incompressible fluid by another. The typical waterflooding process (Craig 1971) for the secondary recovery of oil from petroleum reservoirs is a good example.

Here, in contrast to the previous statistical approaches of network models, we discuss the displacement process when both phases are strictly incompressible in the light of percolation theory. Although a Monte Carlo study can directly account for the incompressibility of both phases, the present study reveals the inadequacy of percolation theory to do so. Nevertheless, it is shown that the theory can be adapted to predict the capillary behavior satisfactorily and without extensive computations. The present work also generalizes the previous applications of percolation theory that have been developed to describe capillary behavior by considering pores with two arbitrarily assigned curvatures (Larson & Morrow 1981) but with varying volume assigned to each of them depending on its size and shape.

We first discuss the basic concepts of percolation theory not only for the convenience of the reader, but also to clarify some of the inconsistencies in the literature with regard to the use of percolation probability.

PERCOLATION THEORY

In general, a percolation model underlying the theory is defined by ascribing a random mechanism to an abstract medium consisting of a collection of points distributed in space. Usually two basic mechanisms known as *bond* and *site* percolations are distinguished (Essam 1980). In the former the points have fixed positions and linkages are made randomly between them. In the latter process, linkages are introduced between points depending on the position of the points.

The original percolation theory problem was advanced by Broadbent & Hammersley (1957) who used graph theory to describe the random mechanism. The points of the medium were considered to be the *vertices* of an infinite graph, the linkages being the *arcs* connecting pairs of vertices. The vertices and arcs are synonymous with sites and bonds respectively. A bond percolation problem was then defined by them by blocking each bond of the medium with a fixed probability q, independently of all other bonds.

In many cases of physical interest the graph L is obtained from a regular space lattice with bonds represented by *edges* that are unordered pairs of vertices. One then easily visualizes the two percolation models (see, e.g. Shante & Kirkpatrick 1971). In the bond percolation problem the random mechanism is introduced by damming the bonds with a fixed probability, $q = 1 - p$, independently of others. "Fluid" then is considered to be blocked when it encounters such a dammed bond and cannot flow through it. In the site problem the stochastic mechanism consists of blocking sites with a fixed probability, $q = 1 - p$, the blocked site chosen at random. Here the fluid cannot pass through a blocked site. Obviously, blocking a site makes all bonds connected to it unfavorable, and fluid cannot traverse through them. The purpose of the theory then is to evaluate the statistical properties of the *clusters* of graph L as a function of p, a cluster being a group of unblocked sites (bonds) linked to one another through unblocked bonds (sites). Since the states of sites or bonds in such problems are categorized into a family of independent two-valued random variables, the above models are called independent or Bernoulli percolation models (Kesten 1982).

An important feature of the above-mentioned problem is the *percolation probability*. The classical definition of this is in terms of random *walks* in graphs (Broadbent & Hammersley 1957; Essam 1972) and is as follows: Given a vertex v, let $S_n(v)$ be the set of n-stepped *self-avoiding walks* from v (Essam & Fisher 1970). In *both* the site and bond problems a walk is said to be open if all its vertices or edges are unblocked. The percolation probability is then defined for the site problem as

$$P^s(v,p) = \lim_{n\to\infty} P_n^s(v,p) \qquad [1]$$

and for the bond problem as

$$P^b(v,p) = \lim_{n\to\infty} P_n^b(v,p) \quad . \qquad [2]$$

Here $P_n^s(v,p)$ and $P_n^b(v,p)$ are the probabilities that at least one of the $S_n(v)$ is open. Clearly, for the graph L under consideration, these probabilities are independent of v and can be redefined as the probability that fluid from a single source site, chosen at random, will wet infinitely many other sites (Frisch & Hammersley 1963; Shante & Kirkpatrick 1971). We then have for both the site and the bond percolation problems that

$$P^j(p) = \lim_{n\to\infty} P_n^j(p), \qquad j=s,b \quad , \qquad [3]$$

where $P_n^j(p)$ is the probability that a single source site wets at least n other sites. An important aspect of the theory then is the existence of the *critical percolation probability* defined as (Frisch & Hammersley 1963)

$$p_c^j = \sup\{p : P^j = 0\} \quad . \qquad [4]$$

Obviously, for $p < p_c^j$ the fluid spreads only locally.

For the bond percolation problem, especially with respect to its application in predicting the capillary behavior the above-mentioned percolation probability, is not of much use if we consider that the pore space is represented by bonds alone. Here it is convenient to define another percolation probability as the probability that fluid from a single source bond wets infinitely many other bonds. We then have

$$\overline{P}^b(p) = \lim_{n\to\infty} \overline{P}_n^b(p) \quad , \qquad [5]$$

where $\overline{P}_n^b(p)$ is the probability that a source bond wets at least n other bonds. A similar definition is possible for the site problem, although we would not find use for it in the following analysis. Unfortunately, not much attention has been paid to the latter definition of percolation probability, and there is no evidence to believe that the two probabilities are the same. [It has been very recently shown by Mason (1984) that these are indeed different for Bethe lattices.]

The infinite site cluster density for the site problem, $R^s(p)$, and the infinite bond cluster density for the bond problem, $\overline{R}^b(p)$, in terms of the percolation probabilities are then given by

$$R^s(p) = p\,P^s(p) \qquad [6]$$

and

$$\overline{R}^b(p) = p\,\overline{P}^b(p) \quad . \qquad [7]$$

Physically, then, $R^s(p)$ [or $R^b(p)$] is the probability that a site (or bond) chosen at random would be connected by an unblocked path to a boundary of sites (or bonds) infinite steps away in a site (or bond) percolation problem.

PORE STRUCTURE

We shall assume that the porous medium is ideal and also sufficiently large. In addition it is presumed that the pore space network is characterized by a representative coordination number that reflects the specific *genus* of the sample. By this we imply that the network can be represented by an *infinite regular lattice graph*, L (Essam & Fisher 1970), the lattice points distributed in two or three dimensions as the case may be.

To each pore identified with a bond or a site of the lattice graph L as described below, we associate two dimensionless curvatures $\mathscr{H}_d$ and $\mathscr{H}_i$ termed drainage and imbibition curvatures. $\mathscr{H}_d$ represents the maximum curvature of the phase interface for the pore and determines the criterion for the nonwetting phase displacing the wetting phase in that pore. Similarly, $\mathscr{H}_i$ is the minimum curvature of the phase interface and determines wetting phase occupancy during imbibition. This presumes that at a given stage of displacement only one phase is allowed to occupy a pore. Our fundamental supposition, then, is that the pore size distribution in terms of $\mathscr{H}_d$ and $\mathscr{H}_i$ is available. Thus the volume probability density function of $(\mathscr{H}_d,\mathscr{H}_i)$ is given as

$$\mathbf{pvdf} = g(\mathscr{H}_d,\mathscr{H}_i) \quad , \qquad [8]$$

where

$$g(\mathscr{H}_d,\mathscr{H}_i) \geq 0, \qquad \mathscr{H}_d > \mathscr{H}_i \qquad [9]$$

and

$$g(\mathscr{H}_d,\mathscr{H}_i) = 0, \qquad \mathscr{H}_d < \mathscr{H}_i \quad . \qquad [10]$$

Correspondingly we write for the number probability density function

$$\mathbf{pndf} = n(\mathscr{H}_d,\mathscr{H}_i) \quad , \qquad [11]$$

where

$$n(\mathscr{H}_d,\mathscr{H}_i) \geq 0, \qquad \mathscr{H}_d > \mathscr{H}_i \qquad [12]$$

and

$$n(\mathscr{H}_d,\mathscr{H}_i) = 0, \qquad \mathscr{H}_d < \mathscr{H}_i \quad . \qquad [13]$$

Both $g(\mathscr{H}_d,\mathscr{H}_i)$ and $n(\mathscr{H}_d,\mathscr{H}_i)$ are assumed continuous. It is also assumed that $(\mathscr{H}_d,\mathscr{H}_i)$ uniquely determine the volume or the inventory of a pore, V_p. This implies that

$$V_p = V(\mathscr{H}_d,\mathscr{H}_i) \quad . \qquad [14]$$

Using that

$$\int_0^\infty \mathrm{d}\mathscr{H}_d \int_0^{\mathscr{H}_d} g(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i = 1 \qquad [15]$$

and

$$\int_0^\infty \mathrm{d}\mathscr{H}_d \int_0^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i = 1 \quad , \qquad [16]$$

we have

$$n(\mathscr{H}_d,\mathscr{H}_i) = K[g(\mathscr{H}_d,\mathscr{H}_i / V(\mathscr{H}_d,\mathscr{H}_i)] \quad , \qquad [17]$$

where

$$K = \left[\int_0^\infty \mathrm{d}\mathscr{H}_d \int_0^{\mathscr{H}_d} \frac{g(\mathscr{H}_d,\mathscr{H}_i)}{V(\mathscr{H}_d,\mathscr{H}_i)} \mathrm{d}\mathscr{H}_i\right]^{-1} \quad . \qquad [18]$$

Alternatively,

$$K = \int_0^\infty \mathrm{d}\mathscr{H}_d \int_0^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i)V(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i \quad . \qquad [19]$$

Thus specification of either $g(\mathscr{H}_d,\mathscr{H}_i)$ or $n(\mathscr{H}_d,\mathscr{H}_i)$ along with V_p implies the knowledge of the other. We shall use $n(\mathscr{H}_d,\mathscr{H}_i)$ in the subsequent analysis for the sake of convenience.

Two possible assignments of pore geometry to the graph L are considered. The first assignment called the *bond representation* is one in which the entire pore is assigned to the bonds with pore geometry. Here the sites have negligible volume, and a phase is continuous across it if more than one bond with that phase is incident to that site. The second case is the *site representation* in which the pore body is assigned to a site, and z equivalent pore throats are assigned to the bonds incident to it, where z is the *degree* of the graph. $\mathscr{H}_i$ is assigned to the pore body and $\mathscr{H}_d$ to the throats. Thus a given bond, being incident to two different sites, would have two different drainage curvatures assigned to it. In other words, the site itself can be assumed to have two curvatures $\mathscr{H}_i$ and $\mathscr{H}_d$ associated with it. A general representation in which these $\mathscr{H}_d$'s are independently assigned would require the site-bond percolation model (Agarwal *et al.* 1979; Hoshen *et al.* 1979) and is not considered here. Finally, since the medium is considered to be chaotic, $\mathscr{H}_i$ and $\mathscr{H}_d$ are randomly assigned to the pores for both of the above-mentioned representations.

HYSTERESIS IN CAPILLARY BEHAVIOR

Saturation changes under quasistatic displacement of one phase by another due to changes in the pressure difference between the two phases, i.e. the capillary pressure, do not follow a unique functional relationship. For the completely wetting/nonwetting system considered here, the contact angle through the wetting phase can be zero, and the hysteresis has been attributed to the familiar ink-bottle effect (Greenkorn 1981) or, equivalently, the irreversible changes associated with capillary displacement (Morrow 1970). Typically, the capillary pressure exhibits the trend shown in figure 1, where the saturation is in terms of the wetting phase with arrows indicating the direction of its change. An important distinction with respect to the two imbibition processes pictured is that one follows complete drainage and the other has an initial nonwetting phase saturation equal to one. It is shown in the following sections that the latter process in general results in a higher residual nonwetting phase saturation. The distinction between the two may then be important especially when one attempts to model the displacement of oil from water-wet reservoirs, where the residual wetting phase is present as connate water.

INITIAL IMBIBITION

We define the initial imbibition process to be one in which the pore space is completely filled with the nonwetting phase and is displaced quasistatically by the wetting phase. The process continues till we attain the residual nonwetting phase at which stage no more capillary displacement is possible without increasing the capillary number (Melrose & Bradner 1974; Ramakrishnan & Wasan 1984). Here the wetting phase occupation occurs due to a reduction in the capillary pressure $\mathscr{P}_{c,I}$, and the following conditions need to be met for its occupancy in a given pore:

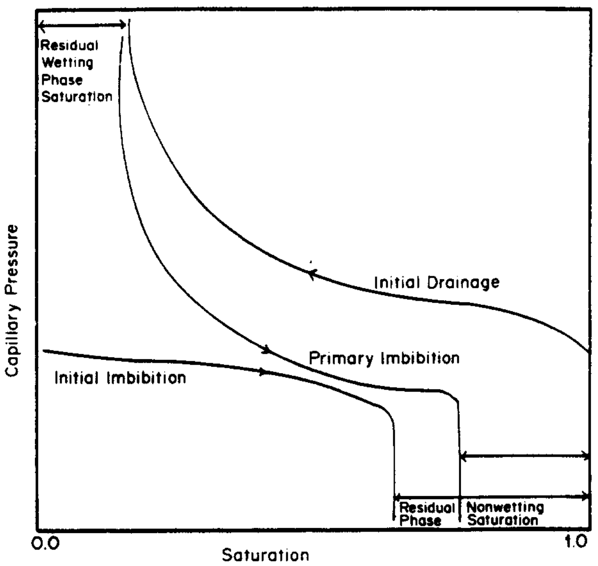


Figure 1. Typical capillary pressure–saturation relationship.

(1) Nonwetting phase can be replaced by the wetting phase in a given pore provided the capillary pressure occupancy criterion is satisfied. The set of pores which satisfy this condition for the prevailing capillary pressure $\mathscr{P}_{c,i}$ is obtained directly from the Laplace jump condition for a phase interface at mechanical equilibrium (Defay & Prigogine 1966) and is

$$\mathscr{P}_{c,i} = \mathscr{H}_i \ , \qquad [20]$$

where $\mathscr{P}_{c,i}$ and $\mathscr{H}_i$ are made appropriately dimensionless (see appendix A). $\mathscr{P}_{c,i}$ is also equal to $\mathscr{P}_{nw} - \mathscr{P}_w$, where $\mathscr{P}_{nw}$ and $\mathscr{P}_w$ are the dimensionless pressures in nonwetting and wetting phases, respectively. Clearly, the set of pores which satisfy this criterion are those with imbibition curvatures greater than $\mathscr{H}_i$ obtained from [20]. These pores are said to be *allowed*, and they constitute the allowed fraction or the *allowability* $p_{w,i}$.

(2) Wetting phase can occupy a pore only if it can "reach" that pore from the boundary of the medium. The approachable fraction or the *approachability* of the invading phase is then defined as the number fraction (based on the entire medium) of the pores which were allowed and could be reached from the boundary of the medium in the absence of condition (3) below. In light of the percolation theory for an infinite lattice, and the assumption of pore network representation as graph L, the approachability is

$$R_{w,i} = R^s(p_{w,i}) \qquad [21]$$

for the site representation, and

$$R_{w,i} = \bar{R}^b(p_{w,i}) \qquad [22]$$

for the bond representation. The approachable fraction then represents physically the number fraction of pores that would be occupied by the wetting phase when the nonwetting phase is infinitely compressible, so that any pore that belongs to the infinite cluster of the allowed fraction would be occupied by the invading phase.

(3) Wetting phase will occupy a pore if the nonwetting phase that is to be displaced from a pore can "reach" the boundary of the medium without passing through the pores occupied by the wetting phase provided conditions (1) and (2) are also met. This implies that the nonwetting phase cannot be displaced if it is surrounded by the wetting phase although both conditions (1) and (2) are satisfied, and it emphasizes the incompressibility of the displaced phase. The number fraction of pores satisfying all three conditions would be termed as the *accessible* fraction. As stressed above, when the displaced phase is infinitely

compressible, there is no distinction between the accessible and the approached fraction. On the contrary, when the displacing phase is infinitely compressible, such as mercury vapor displacing mercury, condition (2) becomes irrelevant. Larson & Morrow (1981) considered this case when $V(\mathscr{H}_d,\mathscr{H}_i) = 1$. An analogous problem also arises when the wetting phase is conducted by the roughness of the pore walls to all of the network and behaves as though it is fully compressible because its allowability becomes equal to its approachability (Lenormand & Zarcone 1984).

Although earlier statistical approaches involving two incompressible fluids have failed to account for all the three conditions discussed, Monte Carlo simulation studies have recognized the necessity of imposing them. For example, explicit use of these conditions have been made by Chandler *et al.* (1982), Lin & Slattery (1982), Mohanty & Salter (1982) and Wilkinson & Willemsen (1983).

Let us now consider any stage of imbibition at which the capillary pressure is $\mathscr{P}_{c,i}$. For a differential reduction in capillary pressure to $\mathscr{P}_{c,i} - \mathrm{d}\mathscr{P}_{c,i}$, $\mathscr{H}_i$ is reduced to $\mathscr{H}_i - \mathrm{d}\mathscr{H}_i$ and the allowability changes to $p_{w,i} + \mathrm{d}\,p_{w,i}$. Denoting approachability as $A_{w,i}$, approachability to newly allowed pores as $A_{w,in}$, and approachability to previously allowed pores as $A_{w,ip}$, we have $\mathrm{d}A_{w,i} = \mathrm{d}A_{w,in} + \mathrm{d}A_{w,ip}$ and

$$\mathrm{d}A_{w,i} = \frac{\mathrm{d}R_{w,i}}{\mathrm{d}p_{w,i}}\,\mathrm{d}p_{w,i} \quad , \tag{23}$$

$$\mathrm{d}A_{w,in} = \frac{R_{w,i}}{p_{w,i}}\,\mathrm{d}p_{w,i} \tag{24}$$

and

$$\mathrm{d}A_{w,ip} = \left(\frac{\mathrm{d}R_{w,i}}{\mathrm{d}p_{w,i}} - \frac{R_{w,i}}{p_{w,i}}\right)\mathrm{d}p_{w,i} \quad . \tag{25}$$

Not all of the incrementally approached pores are accessed because of condition (3).

To estimate the differentially accessed fraction of pores, we first identify two different mechanisms of trapping (Mohanty 1981). The first is the snap-off event in which trapping occurs due to the instability of the wetting phase thin film on the solid surfaces of the pore. The second is the bypass event in which clusters of nonwetting phase are left isolated due to the multiply connected nature of the pore space whereby large $\mathscr{H}_i$ pores enclosing smaller $\mathscr{H}_i$ pores disconnect the nonwetting phase. The two mechanisms have been illustrated in terms of pore doublets by Chatzis *et al.* (1983). For the present we shall consider only the role of the bypass mechanism in the trapping of the nonwetting phase. This is modified later with some assumptions to account for the snap-off events too.

The incrementally accessed fraction is now evaluated by the following arguments. At any capillary pressure, $\mathscr{P}_{c,i} = \mathscr{H}_i$, the two-phase distribution is as pictured in figure 2. All pores with imbibition curvatures $> \mathscr{H}_i$, that constitute the allowed fraction for the wetting phase, can be occupied by both the wetting and nonwetting phases. The wetting phase, being the displacing fluid, has no isolations, and all its pores are connected to the boundary of the sample. Indeed, such pores form a subset of the approachable fraction $R_{w,i}$. The reason that they do not constitute all of $R_{w,i}$ is because of condition (3) mentioned above, whereby some of the pores belonging to $R_{w,i}$ fraction cannot be physically occupied by the wetting phase. These contain nonwetting phase pores which would not escape to the outlet of the sample.

The $p_{w,i}$ allowed fraction of pores, besides containing the wetting phase, can also have nonwetting phase present. This nonwetting phase can be both isolated (in the sense that they can be connected to the boundary of the sample only through wetting phase pores) and nonisolated, for there is no reason to believe otherwise. However, the nonisolated pores in this category necessarily belong to the unapproachable fraction of the wetting phase. Clearly, if they had belonged to the approachable fraction, they could have been replaced by the wetting phase by the very fact that they contain nonisolated nonwetting phase, and all conditions would be met for phase replacement. On the other hand, the isolated regions

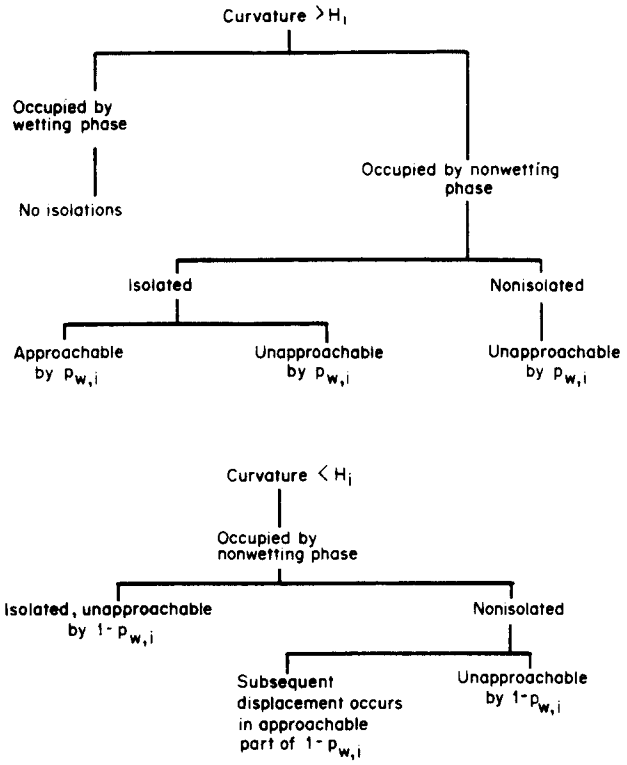


Figure 2. Two-phase distribution at capillary pressure $\mathcal{P}_{c,i} = \mathcal{H}_i$.

of the nonwetting phase could be either approachable or nonapproachable by the wetting phase. A simple illustration of this is presented for the bond representation in the square lattice graph in figure 3. The numbers on the bonds are the imbibition curvatures. From figure 3(a) it is evident that all the pores belong to the approachable fraction of the wetting phase due to the fact that all of them have $\mathcal{H}_i > 5$. In fact, this is also an illustration of a configuration where not all of $R_{w,i}$ is occupied by the wetting phase as mentioned earlier. In figure 3(b) the central bonds with curvatures 19, 18, 22 and 17 (all of them > 5) belong to the unapproachable part of the allowed wetting phase pores, since they are surrounded by pores with curvatures < 5.

Having discussed the distribution of two phases in the allowed wetting phase fraction of pores, we can focus on the category of pores belonging to $1 - p_{w,i}$ fraction. This fraction can be defined to constitute the allowed fraction of the nonwetting phase pores. They form the complement of the pores allowed for the wetting phase in independent percolation (Fisher 1961). For the sake of clarity we will define the approachable part of this fraction to be $R_{nw,i} = R(1 - p_{w,i})$, analogous to $R_{w,i}$.

Pores with curvatures $< \mathcal{H}_i$, that belong to $1 - p_{w,i}$ fraction, cannot be occupied by the wetting phase simply because of condition (1). In subdividing this class of pores we first notice that some of these pores will be occupied by isolated nonwetting phase as described in figure 2. This is because they constitute a *part* of the unapproachable fraction of the nonwetting phase, that is $(1 - p_{w,i}) - R_{nw,i}$. The isolated pores of the nonwetting phase do not constitute all of $(1 - p_{w,i}) - R_{nw,i}$, since some pores with curvatures $> \mathcal{H}_i$ are still occupied by the nonwetting phase. This is clear when one visualizes the two-phase distribution as demonstrated in figure 4(a).

From the above discussion it is evident that the nonisolated part of the nonwetting phase belonging to curvatures $< \mathcal{H}_i$ forms a *boundary* whose *interior* (Broadbent & Hammersley 1957) is partly composed of the nonisolated nonwetting phase pores whose curvatures $> \mathcal{H}_i$. Reiterating our earlier point, these pores with curvatures $> \mathcal{H}_i$ belong to the unapproachable fraction of the wetting phase. The boundary itself is formed by the fraction $R_{nw,i}$, or in a few configurations by $R_{nw,i}$ and $(1 - p_{w,i}) - R_{nw,i}$ as sketched in figures 4(a) and 4(b), respectively. The latter case can arise in rare instances where, in a configuration such as in figure 4(a), some pores belonging to the $(1 - p_{w,i}) - R_{nw,i}$ fraction

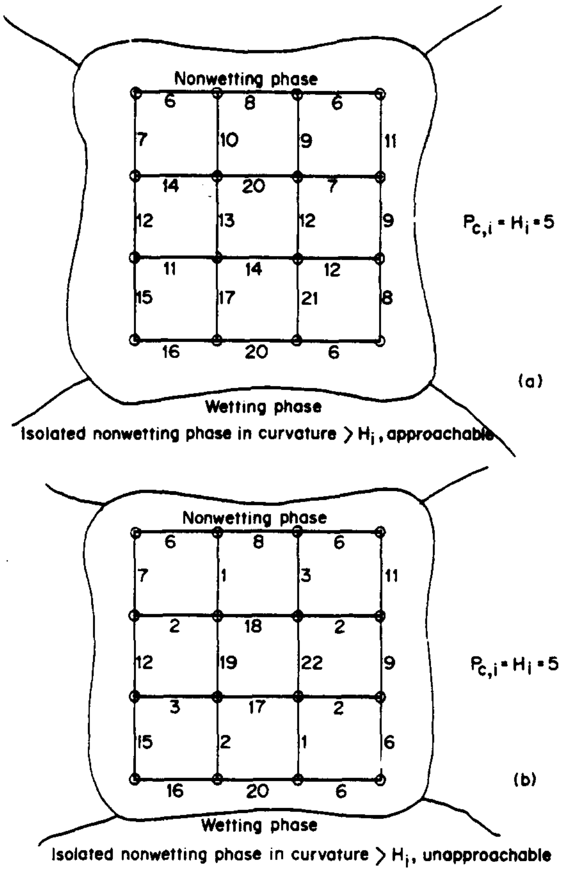
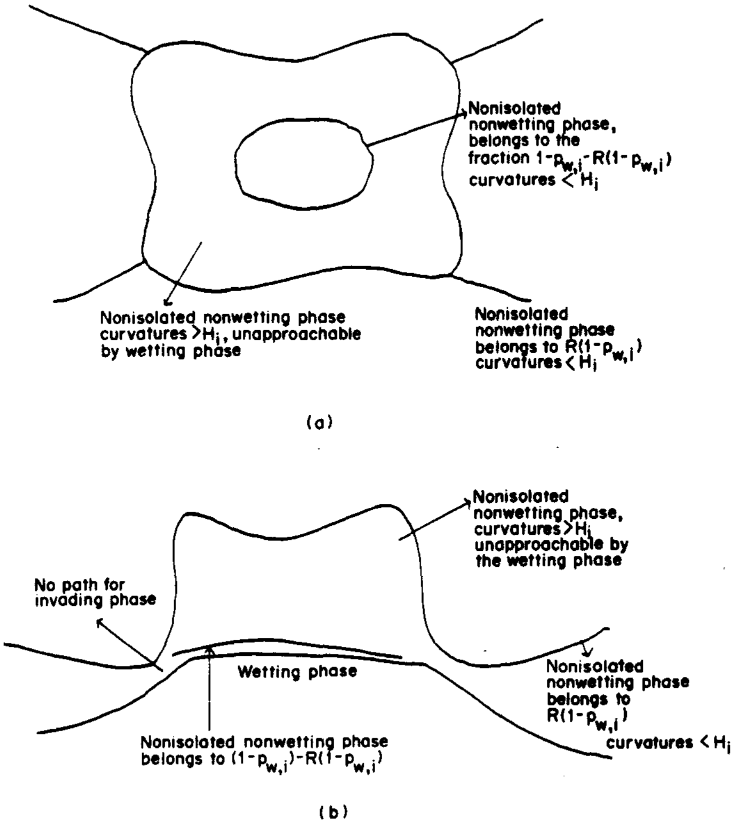


Figure 3. Isolated defender in pores not allowed for it.



ure 4. Schematic sketch of nonisolated, unapproachable nonwetting phase.

are just one step away from the boundary pores, and a subsequent quasistatic invasion in the accessible part of the interior pores can transform the configuration to that of figure 4(b).

With regard to the above description of the two-phase distribution, we make two approximations to obtain differential accessibility, $dC_{w,i}$. The first is that, for every local configuration, the nonisolated nonwetting phase pores belonging to the unapproachable part of $p_{w,i}$ are completely filled before the wetting phase can fill nonisolated nonwetting phase pores belonging to $(1 - p_{w,i}) - R_{nw,i}$, hence isolating the latter class of pores. In the case of figure 4(a) this clearly is a valid assumption, as long as we ignore trapping in pores of curvatures $> \mathscr{H}_i$ where there is a large driving for occupation (see below). For figure 4(b) this is considered to be a good approximation because of two possible reasons: (i) probability of occurrences of these configurations is likely to be small and (ii) even in such configurations, the unapproachable interior pores have a large driving force for occupation, and therefore, chances of their earlier occupation is large. The second approximation we make is that the number ratio of pores that were previously allowed, but approached in isolated and nonisolated regions of the nonwetting phase, is the same as the ratio of freshly approached pores in the respective regions. Also, we assume that the probability of a pore belonging to the $R_{nw,i}$ fraction is the same for both the *freshly* allowed and approached fractions of the wetting phase. Unlike the first assumption, the second one is not a drawback of the classical percolation theory, since it can provide the actual functionality of the fraction of pores in the "holes" with respect to $p_{w,i}$. We adopt this conjecture because of lack of such data, and also because this is the simplest possible assumption consistent with the requirement that the nonwetting phase is totally isolated when $R_{nw,i} = 0$. As in approachability, we split $C_{w,i}$ into two parts $C_{w,in}$ and $C_{w,ip}$ that belong to newly and freshly allowed pores respectively. Thus

$$dC_{w,in} = \frac{R_{w,i}}{p_{w,i}} dp_{w,i} \frac{R_{nw,i}}{(1 - p_{w,i})} \qquad [26]$$

and

$$dC_{w,ip} = \left(\frac{dR_{w,i}}{dp_{w,i}} - \frac{R_{w,i}}{p_{w,i}}\right) dp_{w,i} \frac{R_{nw,i}}{(1 - p_{w,i})} \quad . \qquad [27]$$

The distinction between the two is important if one is interested in inventory changes rather than in number fractions. In fact, the invasion of the wetting phase into pores belonging to $dC_{w,ip}$ is an irreversible process and takes place in the form of Haines jumps (Haines 1930; Morrow 1970) and are referred to as hygrons (Melrose & Bradner 1974). In finite samples this causes the "jagged" character of the capillary pressure curves, although by repeated trials this would be smoothed out in the computer simulation of wetting phase invasion (Androutsopoulos & Mann 1979; Mann *et al.* 1981). In infinite samples, medium saturation changes would be infinitesimally small for differential changes in allowability, although local saturations can be expected to change discontinuously.

To obtain the differential inventory changes in terms of the saturation of the wetting phase $S_{w,i}$, consider the set of pores with curvatures between $\mathscr{H}_d$ and $\mathscr{H}_d + d\mathscr{H}_d$, that is, for the present we follow saturation changes only in pores with this range of drainage curvatures. The differential change in saturation due to freshly allowed pores, ignoring the thin film inventory, is

$$\frac{\partial^2 S_{wi,n}}{\partial \mathscr{H}_i \partial \mathscr{H}_d} d\mathscr{H}_d \, d\mathscr{H}_i = n(\mathscr{H}_d,\mathscr{H}_i) \, d\mathscr{H}_d \, d\mathscr{H}_i \frac{R_{nw,i}}{(1 - p_{w,i})} \frac{R_{w,i}}{p_{w,i}} \frac{V(\mathscr{H}_d,\mathscr{H}_i)}{K} \quad . \qquad [28]$$

The corresponding change in the previously allowed pores for a chaotic medium is

$$\frac{\partial^2 S_{wi,p}}{\partial \mathscr{H}_i \partial \mathscr{H}_d} d\mathscr{H}_d \, d\mathscr{H}_i = n(\mathscr{H}_d,\mathscr{H}_i) \, d\mathscr{H}_i \frac{R_{nw,i}}{(1 - p_{w,i})} \left(\frac{dR_{w,i}}{dp_{w,i}} - \frac{R_{w,i}}{p_{w,i}}\right) \qquad [29]$$

$$\cdot \frac{\int_{\mathscr{H}_i}^{\infty} n(\mathscr{H}_d, \mathscr{H}_i) \, V(\mathscr{H}_d, \mathscr{H}_i) \, d\mathscr{H}_i}{K \int_{\mathscr{H}}^{\infty} n(\mathscr{H}_d,\mathscr{H}_i) \, d\mathscr{H}_i} \; .$$

Equations [28] and [29] can then be summed and integrated with respect to $\mathscr{H}_d$ to get

$$\mathrm{d}S_{w,i} = \left\{ \int_0^\infty n(\mathscr{H}_d,\mathscr{H}_i) \frac{R_{nw,i}}{K(1-p_{w,i})} \left[\left(\frac{\mathrm{d}R_{w,i}}{\mathrm{d}p_{w,i}} - \frac{R_{w,i}}{p_{w,i}} \right) \cdot \frac{\int_{\mathscr{H}_i}^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i) V(\mathscr{H}_d,\mathscr{H}_i)\, \mathrm{d}\mathscr{H}_i}{\int_{\mathscr{H}_i}^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i)\, \mathrm{d}\mathscr{H}_i} + \frac{R_{w,i}}{p_{w,i}} V(\mathscr{H}_d,\mathscr{H}_i) \right] \mathrm{d}\mathscr{H}_d \right\} \mathrm{d}\mathscr{H}_i \ . \qquad [30]$$

From [30], the saturation at the prevailing capillary pressure $\mathscr{P}_{c,i}$ is easily obtained by integration. The result is

$$S_{w,i} = \int_{\mathscr{P}_{c,i}}^\infty \mathrm{d}\mathscr{H}_i \int_{\mathscr{H}_i}^\infty \left[n(\mathscr{H}_d,\mathscr{H}_i) \frac{R_{nw,i}}{K(1-p_{w,i})} \left\{ \left(\frac{\mathrm{d}R_{w,i}}{\mathrm{d}p_{w,i}} - \frac{R_{w,i}}{p_{w,i}} \right) \cdot \frac{\int_{\mathscr{H}_i}^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i) V(\mathscr{H}_d,\mathscr{H}_i)\, \mathrm{d}\mathscr{H}_i}{\int_{\mathscr{H}_i}^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i)\, \mathrm{d}\mathscr{H}_i} + \frac{R_{w,i}}{p_{w,i}} V(\mathscr{H}_d,\mathscr{H}_i) \right\} \right] \mathrm{d}\mathscr{H}_d \ , \qquad [31]$$

with

$$p_{w,i} = \int_{\mathscr{P}_{c,i}}^\infty \mathrm{d}\mathscr{H}_i \int_{\mathscr{H}_i}^\infty n(\mathscr{H}_d,\mathscr{H}_i)\, \mathrm{d}\mathscr{H}_d \ , \qquad [32]$$

where $\mathscr{P}_{c,i}$ is the prevailing capillary pressure. Obviously the process ends when $R_{nw,i} = 0$ or $1 - p_{w,i} = p_c^j$. This is the stage when the residual nonwetting phase saturation is attained, and no more nonwetting phase displacement is possible.

INITIAL DRAINAGE

The initial drainage process begins with a quasistatic displacement of the wetting phase wholly occupying the pore space of the medium, by the nonwetting phase, through a gradual increase of the capillary pressure, $\mathscr{P}_{c,d}$. Examples of this process are oil injection in water-wet sandstones or mercury injection into a sample filled with water. During this process we impose the following conditions to be satisfied for the displacement of the wetting phase in any pore:

(1) Analogous to the imbibition process, the fraction of pores allowed to be occupied at a capillary pressure $\mathscr{P}_{c,d}$ have drainage curvatures less than

$$\mathscr{H}_d = \mathscr{P}_{c,d} \ . \qquad [33]$$

This constitutes the allowed fraction of pores equal to $p_{nw,d}$.

(2) Nonwetting phase can occupy a pore only if it can "reach" that pore from the boundary of the medium. The set of pores satisfying this condition would then be the approachable fraction given by

$$R_{nw,d} = R^s(p_{nw,d}) \qquad [34]$$

or

$$R_{nw,d} = \overline{R}^b(p_{nw,d}) \ , \qquad [35]$$

as the case may be.

(3) Due to the presence of thin film states of the wetting phase on the pore walls, we consider that the wetting phase will be replaced from a pore as long as an infinite cluster of the wetting phase is present in the medium provided that the above conditions are also met. When no infinite cluster is present we assume that the overall resistance to flow through thin film states is so large that no more displacement occurs.

From the above conditions, the drainage process proceeds whereby the approachable fraction of the nonwetting phase would be permitted occupation till $R(1 - p_{nw,d}) = R_{w,d} = 0$, that is, when $1 - p_{nw,d} = p_c^j$. The allowability for the nonwetting phase, using [13] is

$$p_{nw,d} = \int_0^{\mathscr{P}_{c,d}} \mathrm{d}\mathscr{H}_d \int_0^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i \quad . \qquad [36]$$

The number fraction of pores occupied by the nonwetting phase is $R_{nw,d}$ and, therefore, that of the wetting phase is $1 - R_{nw,d}$. The number probability function of pores occupied by the wetting phase ($n_{w,d}$) is necessary in order to compute the fractional volume occupied by it. For a medium with random assignment of pores, this is

$$n_{w,d}(\mathscr{H}_d,\mathscr{H}_i) = \begin{cases} \dfrac{(p_{nw,d} - R_{nw,d})}{p_{nw,d}(1 - R_{nw,d})}\, n(\mathscr{H}_d,\mathscr{H}_i), \ \forall\, \mathscr{H}_d < \mathscr{P}_{c,d} \ ; \\[2ex] \dfrac{1}{(1 - R_{nw,d})}\, n(\mathscr{H}_d,\mathscr{H}_i), \ \forall\, \mathscr{H}_d > \mathscr{P}_{c,d} \ . \end{cases} \qquad [37]$$

The saturation of the wetting phase for any capillary pressure $\mathscr{P}_{c,d}$ is

$$S_{w,d} = [1 - R_{nw,d}]\,\frac{\displaystyle\int_0^{\infty} \mathrm{d}\mathscr{H}_d \int_0^{\mathscr{H}_i} n_{w,d}(\mathscr{H}_d,\mathscr{H}_i)\, V(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i}{\displaystyle\int_0^{\infty} \mathrm{d}\mathscr{H}_d \int_0^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i) V(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i} \qquad [38]$$

or, from [37],

$$S_{w,d} = \frac{p_{nw,d} - R_{nw,d}}{K p_{nw,d}} \left[\int_0^{\mathscr{P}_{c,d}} \mathrm{d}\mathscr{H}_d \int_0^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i) V(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i\right] + \frac{1}{K}\int_{\mathscr{P}_{c,d}}^{\infty} \mathrm{d}\mathscr{H}_d \int_0^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i) V(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i \ , \qquad [39]$$

with $R_{nw,d}$ and $p_{nw,d}$ determined from $\mathscr{P}_{c,d}$ through [34] or [35] and [36]. The expression for the residual saturation S^0_{rw} is obtained from the condition that drainage ceases when no infinite wetting phase clusters are present, or, $p_{nw,d} = 1 - p_c^j$. Thus

$$S^0_{rw} = \frac{1}{K}\left(\frac{(1 - p_c^j) - R_{nw,d}(1 - p_c^j)}{(1 - p_c^j)}\right) \cdot \left[\int_0^{\mathscr{H}_{dc}} \mathrm{d}\mathscr{H}_d \int_0^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i) V(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i\right] + \frac{1}{K}\int_{\mathscr{H}_{dc}}^{\infty} \mathrm{d}\mathscr{H}_d \int_0^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i)\, V(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i \ , \qquad [40]$$

where $\mathscr{H}_{dc}$ is obtained from

$$1 - p_c^j = \int_0^{\mathscr{H}_{dc}} \mathrm{d}\mathscr{H}_d \int_0^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i \quad . \qquad [41]$$

A useful approximation that sacrifices little numerical accuracy is possible for many lattices with p_c^j substantially smaller than $\frac{1}{2}$ (Heiba *et al.* 1982). For such lattices

$$R_{nw,d}\,(1 - p_c^j) \approx 1 - p_c^j \quad . \tag{42}$$

Thus eq. [40] readily simplifies to

$$S_{rw}^0 = \frac{1}{K}\int_{\mathscr{H}_{dc}}^{\infty} \mathrm{d}\mathscr{H}_d \int_0^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i)V\,(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i \quad , \tag{43}$$

implying that the residual wetting phase saturation, or the connate water (with water as the wetting phase) in petroleum reservoir terminology, is contained in pores with drainage curvatures $> \mathscr{H}_{dc}$.

PRIMARY IMBIBITION

The term primary imbibition refers to the displacement of the nonwetting phase after the completion of drainage. This is of practical relevance in water-wet reservoirs when the recovery of oil occurs under capillary controlled conditions (Craig 1971).

The initial drainage analysis shows that under conditions when $p_c^j \geq \frac{1}{2}$, the residual saturation in infinite media is one. Obviously this case is of no relevance in regard to the study of primary imbibition. We shall consider only those cases where S_{rw}^0 is obtained by using [43]. Here the residual wetting phase occupies all pores with drainage curvatures $> \mathscr{H}_{dc}$. The number fraction of these pores is p_c^j and these pores occupy random positions in the medium.

With a quasistatic decrease in capillary pressure the imbibition curvature which determines the allowability criterion is gradually reduced. Contrary to initial imbibition, however, defining allowability based on the prevailing capillary pressure alone leads to insurmountable difficulties. Since the residual wetting phase removal from pores with $\mathscr{H}_d > \mathscr{H}_{dc}$ is clearly not possible (this would amount to drainage), it is easier to solve this problem by redefining allowability where all pores with $\mathscr{H}_d > \mathscr{H}_{dc}$ are permitted to be occupied by the wetting phase. Thus the allowability for the wetting phase at a capillary pressure $\mathscr{P}_{c,i} < \mathscr{H}_{dc}$ is given by

$$p_{w,i} = p_{w,i1} + \int_0^{\mathscr{P}_{c,i}} n(\textstyle\int_{\mathscr{H}_{dc}}^{\infty},\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i \quad , \tag{44}$$

where analogous to [32]

$$p_{w,i1} = \int_{\mathscr{P}_{c,i}}^{\infty} \mathrm{d}\mathscr{H}_i \int_{\mathscr{H}_i}^{\infty} n(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_d \tag{45}$$

and

$$n(\textstyle\int_{\mathscr{H}_{dc}}^{\infty},\mathscr{H}_i) = \displaystyle\int_{\mathscr{H}_{dc}}^{\infty} n(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i \quad . \tag{46}$$

Conditions (2) and (3) described in initial imbibition remain the same with the approachability defined with respect to the modified allowability.

When an excess differential allowability is apportioned to the wetting phase by decreasing the capillary pressure, excess approachability beyond that of freshly allowed pores can be classified into the following subsets:

(1) Pores of nonwetting phase enclosed by $R(1 - p_{w,i})$ and possibly $1 - p_{w,i} - R\,(1 - p_{w,i})$ fraction. These are accessed.

(2) Pores of isolated nonwetting phase enclosed by $(1 - p_{w,i}) - R(1 - p_{w,i})$ fraction. These are not accessed.

(3) Pores of the wetting phase that initially comprised the residual saturation. Here, no subclassification like the above is really necessary.

In measuring the saturation change we are not interested in the pores of category (2) and (3). Based on the arguments similar to that of the initial imbibition section and figure 5 which presents the modified two-phase distribution for primary imbibition it is evident that $\mathrm{d}A_{w,ip}$ is constituted by categories (1) and (2). Then we have that

$$\mathrm{d}A_{w,ip} = \left(\frac{\mathrm{d}R_{w,i}}{\mathrm{d}p_{w,i}} - \frac{R_{w,i}}{p_{w,i}}\right)\left(\frac{p_{w,i} - p_c^j}{p_{w,i}}\right)\mathrm{d}p_{w,i} \quad . \qquad [47]$$

The redefinition of allowability as in [44] eliminates category (3) in the above equation. As before,

$$\mathrm{d}A_{w,in} = \frac{R_{w,i}}{p_{w,i}}\,\mathrm{d}p_{w,i} \quad . \qquad [48]$$

Analogous to [26] and [27], the differential accessibilities would be

$$\mathrm{d}C_{w,in} = \frac{R_{w,i}}{p_{w,i}}\,\mathrm{d}p_{w,i}\,\frac{R_{nw,i}}{(1 - p_{w,i})} \qquad [49]$$

and

$$\mathrm{d}C_{w,ip} = \left(\frac{\mathrm{d}R_{w,i}}{\mathrm{d}p_{w,i}} - \frac{R_{w,i}}{p_{w,i}}\right)\left(\frac{p_{w,i} - p_c^j}{p_{w,i}}\right)\mathrm{d}p_{w,i}\,\frac{R_{nw,i}}{(1 - p_{w,i})} \quad . \qquad [50]$$

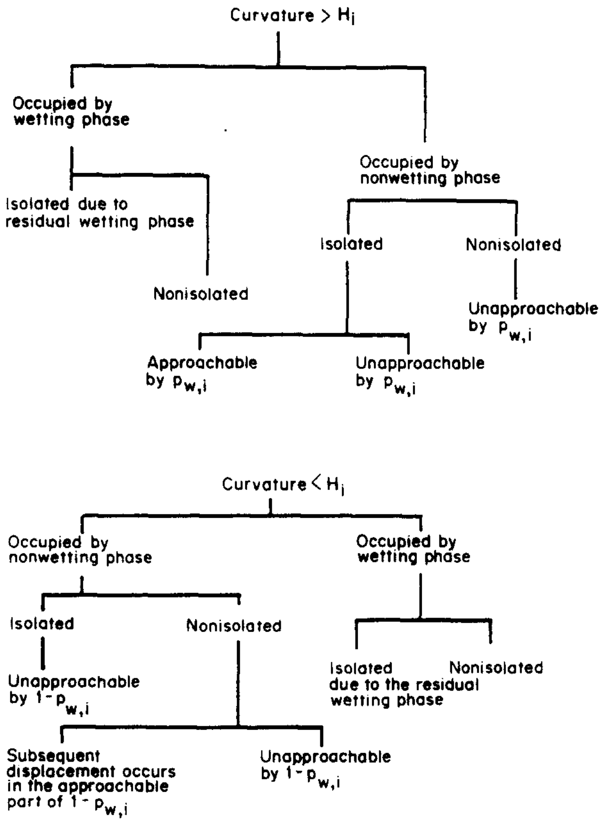


Figure 5. Two-phase distribution at $\mathscr{P}_{c,i} = \mathscr{H}_i$ in primary imbibition.

Inventory changes are now obtained as before by considering the fractional volume changes in pores with curvatures $\mathscr{H}_d$ to $\mathscr{H}_d + \mathrm{d}\mathscr{H}_d$ due to the differential accessibility. This is

$$\frac{\partial^2 S_{w,i}}{\partial \mathscr{H}_i \partial \mathscr{H}_d}\,\mathrm{d}\mathscr{H}_d\,\mathrm{d}\mathscr{H}_i = n(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_d\,\mathrm{d}\mathscr{H}_i(1 - H(\mathscr{H}_d - \mathscr{H}_{dc}))\,\frac{R_{nw,i}}{K(1-p_{w,i})} \quad [51]$$

$$\cdot\left[\left(\frac{\mathrm{d}R_{w,i}}{\mathrm{d}p_{w,i}} - \frac{R_{w,i}}{p_{w,i}}\right)\left(\frac{p_{w,i} - p_c^j}{p_{w,i}}\right)\right.$$

$$\left.\cdot\frac{\int_{\mathscr{H}_i}^{\mathscr{H}_{dc}} n(\mathscr{H}_d,\mathscr{H}_i)V(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i}{\int_{\mathscr{H}_i}^{\mathscr{H}_{dc}} n(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i} + \frac{R_{w,i}}{p_{w,i}}V(\mathscr{H}_d,\mathscr{H}_i)\right],$$

where H is the Heaviside function. Integrating with respect to $\mathscr{H}_d$ and $\mathscr{H}_i$ and using the condition at the completion of drainage namely, $\mathscr{H}_i = \mathscr{H}_{dc}$, $S_w = S_{rw}^0$, we have the result that

$$S_{w,i} = S_{rw}^0 + \int_{\mathscr{P}_{c,i}}^{\mathscr{H}_{dc}} \mathrm{d}\mathscr{H}_i \int_{\mathscr{H}_i}^{\mathscr{H}_{dc}} n(\mathscr{H}_d,\mathscr{H}_i)\frac{R_{nw,i}}{K(1-p_{w,i})}\left[\left(\frac{\mathrm{d}R_{w,i}}{\mathrm{d}p_{w,i}} - \frac{R_{w,i}}{p_{w,i}}\right)\right.$$

$$\left.\cdot\left(\frac{p_{w,i} - p_c^j}{p_{w,i}}\right)\frac{\int_{\mathscr{H}_i}^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i)V(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i}{\int_{\mathscr{H}_i}^{\mathscr{H}_d} n(\mathscr{H}_d,\mathscr{H}_i)\,\mathrm{d}\mathscr{H}_i} + \frac{R_{w,i}}{p_{w,i}}V(\mathscr{H}_d,\mathscr{H}_i)\right]\mathrm{d}\mathscr{H}_d \ . \quad [52]$$

Again the imbibition process ends when $1 - p_{w,i} = p_c^j$.

SNAP-OFF PROCESS

The snap-off or the choke-off event occurs due to the instability of the thin film states of the wetting phase (Mohanty 1981). To account for this event Mohanty & Salter (1982) used the condition that when

$$\frac{\mathscr{H}_d}{\mathscr{H}_i} > M \ , \quad [53]$$

where M is some critical value ≈ 3, choke-off can occur. This choke-off, for example, in imbibition, prevents the complete occupation of the wetting phase in a pore, although all conditions of displacement are met. In the foregoing analysis it is possible to account for this event through some simple justifications. Only the initial imbibition process is considered.

We assume that choke-off occurs only when a nearest neighbor pore contains the wetting phase, and only when the capillary condition is satisfied. Then the snap-off event simply prevents the nonwetting phase removal from a pore, since it isolates it. In such a situation the inventory of the wetting phase in that pore can be considered to be negligible, and the process can be described through [31] with the condition that

$$V(\mathscr{H}_d,\mathscr{H}_i) = 0,\ \forall\frac{\mathscr{H}_d}{\mathscr{H}_i} \geq M = 3 \ . \quad [54]$$

PREDICTION OF CAPILLARY BEHAVIOR

Equations [31], [39] and [52] can now be used to predict the capillary pressure–saturation relationship for the three processes discussed, provided that the graph L, the curvature distribution and the pore geometry are known *a priori*. Since detailed photomicrographic information is currently unavailable, only qualitative comparisons between theoretical calculations with appropriate choice of structural data and experiments can be made.

The graph L, chosen for the purpose of obtaining the percolation probabilities, was the Bethe lattice or the infinite Cayley tree of degree z. It is an endlessly branching graph and is acyclic, that is, two vertices of such a graph are connected by a unique path (Essam & Fisher 1970). A Bethe lattice with $z = 4$ is shown in figure 6. The choice of the Bethe lattice in no way implies the pore structure in terms of the network representation, and clearly the discussion of the imbibition process presented in this study is invalid for the Bethe lattice because the presence of allowed but unapproachable pores of the wetting phase can significantly affect the accessible fraction in these lattices.

We use the tree as the graph because analytical expressions are possible for the various percolation properties of such lattices, while the commonly encountered lattice graphs in problems of physical interest require the use of Monte Carlo data. A comparison of the infinite cluster density $R^s(p)$ of the Bethe lattice with that of the face centered cubic (FCC) and the simple cubic (SC) lattices in figure 7, indicate that, barring the critical behavior, the Bethe lattice is adequate for predicting the capillary behavior. This is because only the approachability functions of the lattice enter the final equations that determine the capillary pressure–saturation relationship. Thus, owing to the use of the Bethe lattice, z is the only parameter of interest regarding lattice type. $R^s(p)$ and $\overline{R}^b(p)$ for Bethe lattices were obtained by Fisher & Essam (1961) from generating functions and are

$$R^s(p) = p - p^*(1 - p)^2/(1 - p^*)^2, \; p > p^j_c \quad , \qquad [55]$$

$$\overline{R}^b(p) = p\left\{ 1 - \left(\frac{p^*}{p}\right)^{\frac{(2z-2)}{(z-2)}} \right\}, \quad p > p^j_c \quad , \qquad [56]$$

where

$$p^*(1 - p^*)^{z-2} = p(1 - p)^{z-2} \quad . \qquad [57]$$

Also, for, $p < p^j_c = 1/(z - 1)$

$$R^s(p) = \overline{R}^b(p) = 0 \quad . \qquad [58]$$

The root of [57] vanishes continuously with $p(1 - p)^{z-2}$.

A slightly modified form of the probability density function presumed by Larson & Morrow (1981) is used for $g(\mathscr{H}_d,\mathscr{H}_i)$. This is

$$g(\mathscr{H}_d,\mathscr{H}_i) = \begin{cases} 48k^3 f(\mathscr{H}_d) f(\mathscr{H}_i) \displaystyle\int_{\mathscr{H}_i}^{\mathscr{H}_d} f(\mathscr{H})\,\mathrm{d}\mathscr{H}, \; \mathscr{H}_d > \mathscr{H}_i \; ; \\ 0, \; \mathscr{H}_d < \mathscr{H}_i \quad , \end{cases} \qquad [59]$$

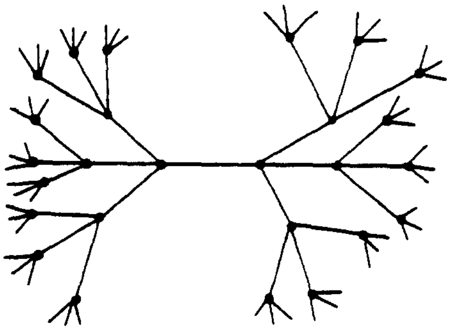


Figure 6. Bethe lattice with $z = 4$.

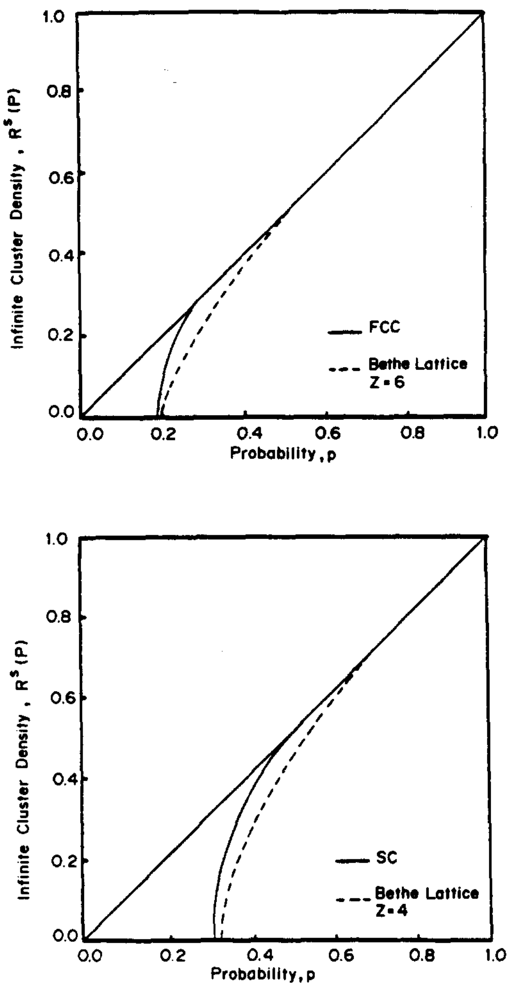


Figure 7. Infinite cluster densities in bethe, FCC and SC Lattices.

with

$$f(\mathcal{H}) = \mathcal{H} e^{-k\mathcal{H}^2} , \qquad [60]$$

where k is a parameter that determines both the mean and the variance of the density function.

Three types of pore geometry for the bond representation were considered. These were the sinusoidal pore, the rectangular pore approaching parallel plates and the simple bond pore. The simple bond pore system is one where all the pores are assigned the same volume irrespective of $(\mathcal{H}_d, \mathcal{H}_i)$ and is useful in making comparisons with the independent percolation results. Only the simple site assignment was considered for the site representation problem. Volume assignment and the Laplace jump conditions for these pore geometries in terms of dimensionless variables ([20] and [33]) are presented in appendix B.

The computation of saturation in terms of the capillary pressure requires in general, the evaluation of multiple integrals. A Newton-Cotes adaptive eight-point quadrature (Forsythe *et al.* 1977) with recursive calling of the integration subroutine was developed for this purpose. The critical curvatures that correspond to the end of drainage or imbibition were evaluated using a generalized Newton-Raphson algorithm.

RESULTS AND DISCUSSION

The dependence of the predicted capillary behavior on k for all three processes is shown in figure 8 for the sinusoidal pore geometry. From this figure and the form of $g(\mathcal{H}_d, \mathcal{H}_i)$ it is evident that changing the value of k only shifts the mean p^j_c and alters the

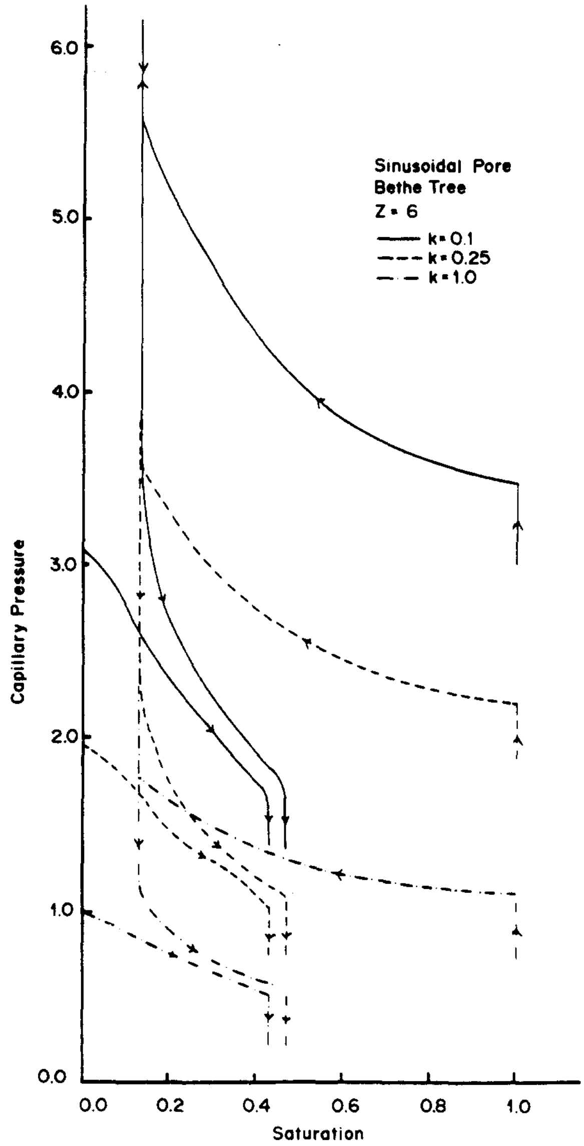


Figure 8. Capillary pressure curves with changing k.

spread of the capillary pressure curve. As expected, changing the spread of the distribution has no effect on the residual saturation of either of the phases in any of the processes. These results obviously hold for other geometries as well.

Changing the pore geometry does not shift the position of the capillary pressure curves appreciably (see figure 9). The threshold pressures at which saturations in initial drainage and imbibition begin to change would be affected because the pvdf is fixed. With changing volume assignment to the pores the number fraction equals p_c^j at different curvatures. With increased allowability however, $\overline{R}^b(p) \rightarrow p$, and a universal pvdf implies that the capillary pressure characteristics would be the same. The residual saturations are affected to a large degree, again because of the volume assignment. For example, when the drainage process is considered, given a number fraction p_c^j at which the wetting phase ceases to be replaced, a smaller volume fraction is assigned to it in the case of sinusoidal pores than for rectangular or simple bond pores. The reverse argument is true for the imbibition process. As seen in figure 9, the sinusoidal pore geometry results in a larger amount of the trapped nonwetting phase. Therefore, pore geometry is a parameter of influence in determining residual saturations. Any interpretation of the coordination number z in terms of the residual saturation (Yuan 1981) needs to consider this.

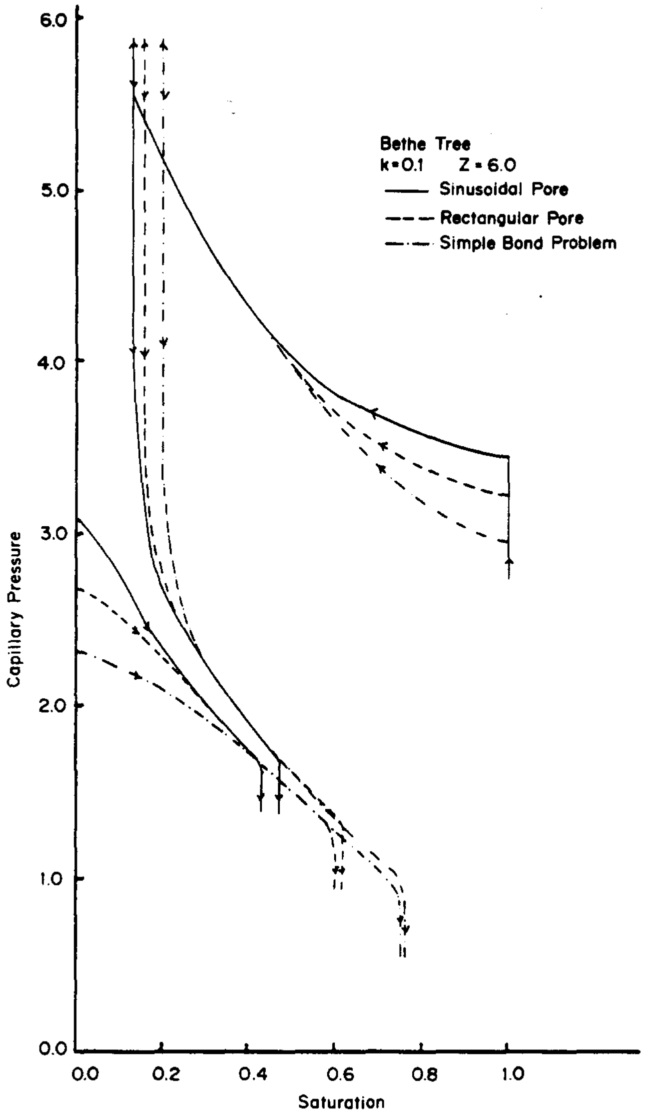


Figure 9. The effect of pore geometry on the capillary-pressure saturation relationship.

The initial and the primary imbibition process can result in different amounts of residual nonwetting phases. The difference is noticeable especially for a low coordination number lattice and a pore geometry which enhances the number fraction of pores in conversion to volume fraction. The plots of figure 10 where the influence of z is studied for the sinusoidal pore confirms this argument. Here for a $z = 4$, the difference in the residual saturations is significant. Figures 11 and 12 are similar plots for the rectangular pore and the simple bond pore. In the latter case the residual saturations in the two imbibition processes are virtually identical even for $z = 4$.

The simple site problem where the pores are assigned equal inventory with site representation produces very similar results for the capillary pressure characteristics (figure 13). The residual wetting phase saturation is identical to that of the simple bond problem, although the same is not true for the nonwetting phase in spite of the fact that $p_c^s = p_c^b$. The site problem yields a larger residual saturation, and also the difference in the residual saturation is significant for a small z.

The influence of the pore geometry and z on the residual saturations is shown in figures 14–16. Obviously, since p_c^b decreases with z, residual saturations follow the same trend. For the sinusoidal and the rectangular pores the residual saturations are significantly different from the critical percolation probability as discussed above. For the simple bond or site

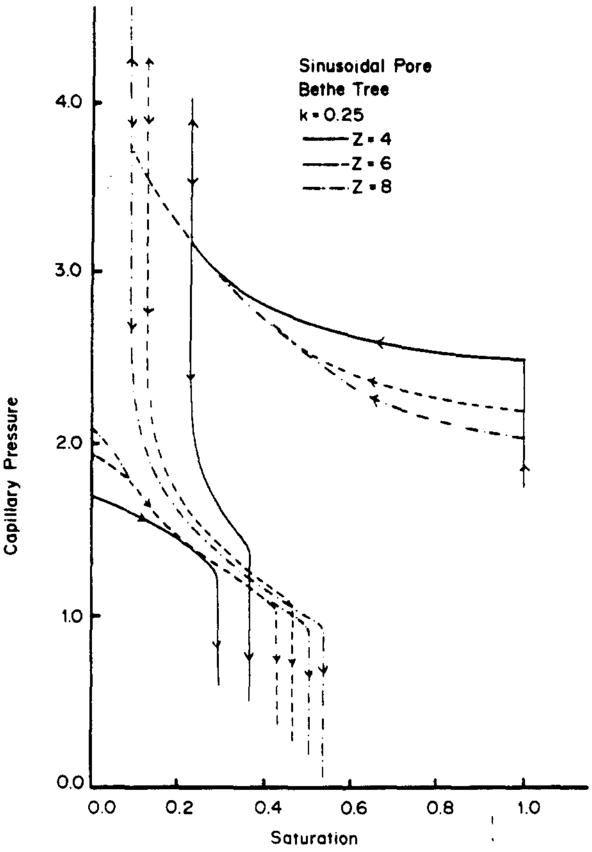


Figure 10. The effect of coordination number on the capillary behavior for sinusoidal pores.

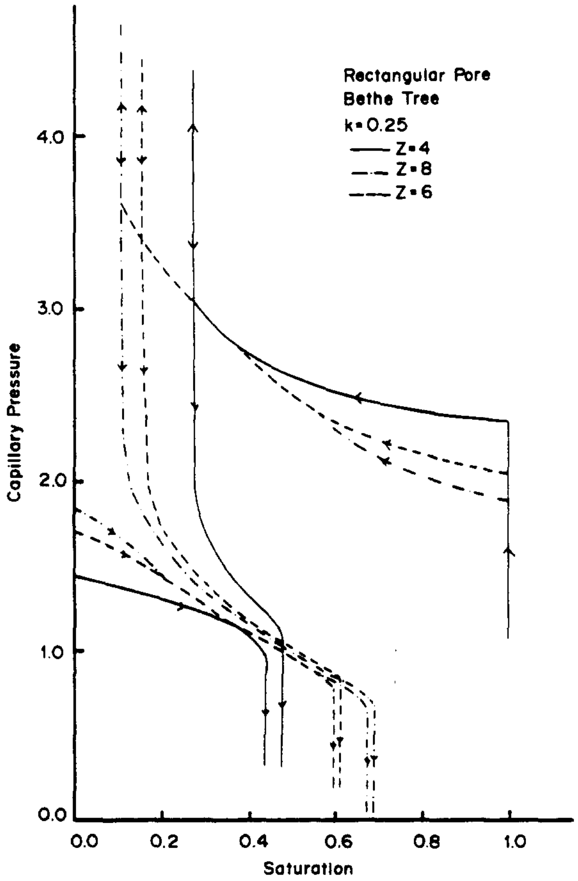


Figure 11. The effect of coordination number on the capillary behavior for rectangular pores.

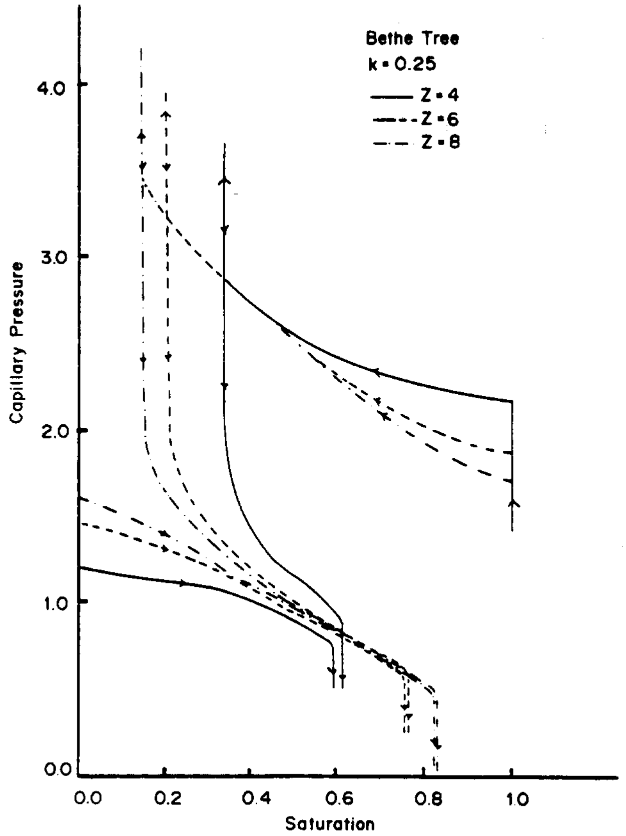


Figure 12. The effect of coordination number on the capillary behavior for the simple bond problem.

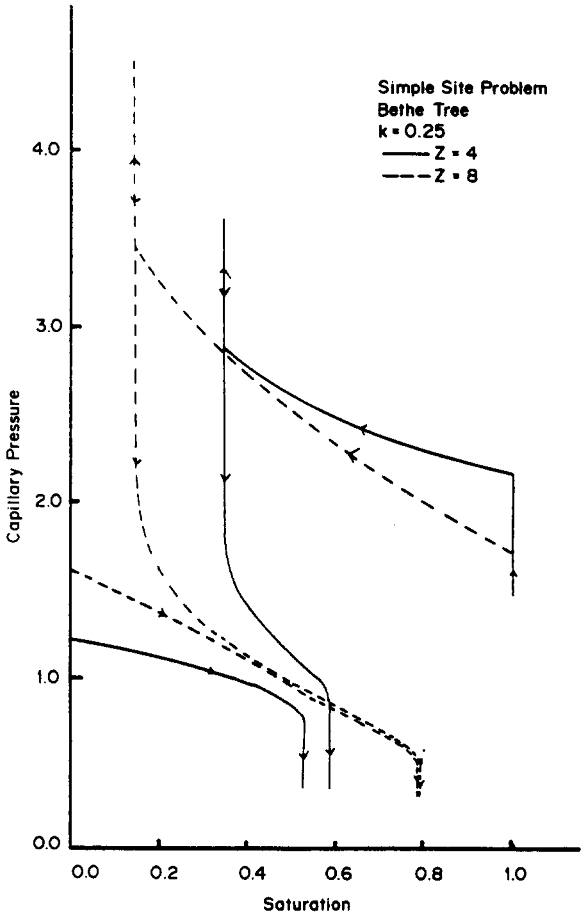


Figure 13. Capillary behavior for the simple site problem.

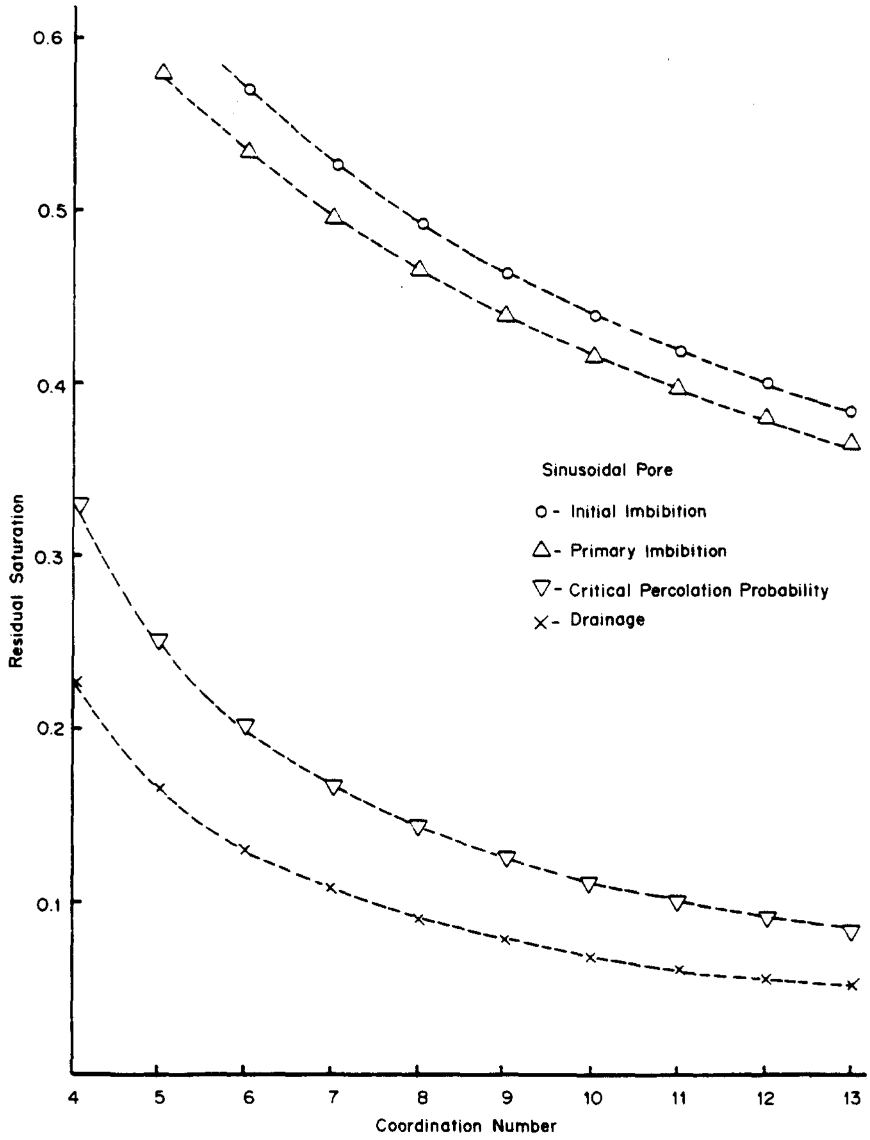


Figure 14. Residual saturations for the sinusoidal pore.

problems, it is evident from [43] that the residual wetting phase saturation is identical to p^j_c. The residual nonwetting phase saturation, however, is greater than the critical percolation probability. This result is different from the commonly employed assumption (deGennes & Guyon 1978; Larson *et al.* 1981) that the residual number fraction $= p^j_c$, because this assumption ignores the condition that in no finite cluster of the nonwetting phase can a phase replacement occur. It remains to be seen whether the critical exponents are affected by this difference.

A modified form of percolation known as invasion percolation has recently been suggested by Wilkinson & Willemson (1983). Here the nonreplaceability of the isolated phase can be explicitly taken into account. Invasion percolation, however, is slightly different from the phase replacement process studied here because (i) in the former process, at a given step, only the least resistant pore is considered for replacement. When the trapping of the displaced phase is taken into account, then the invaded phase undergoes phase replacement only if it belongs to the infinite cluster of the displaced phase, and (ii) a specific injection face is identified in invasion percolation.

The effect of the snap-off events on capillary behavior is illustrated in figure 17 for the simple bond problem for $z = 4$ and 6. Clearly, the wetting phase thin film instability alters the capillary pressure curve noticeably and increasingly with increasing z. This is because, in the absence of the snap-off event, the residual nonwetting phase saturation

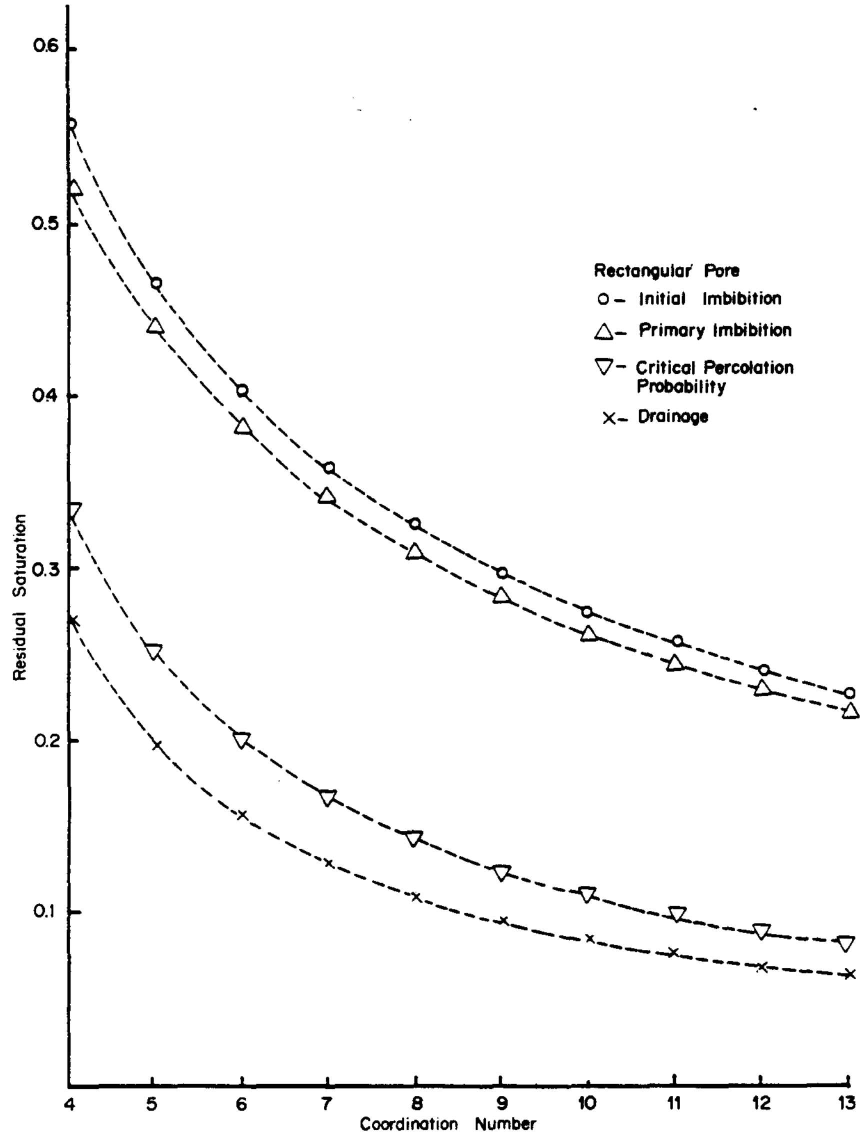


Figure 15. Residual saturations for the rectangular pore.

approaches 0 with $z \to \infty$, whereas in the presence of this event there would be a nonzero residual nonwetting phase saturation, its value depending on the pore size distribution. In the case of axisymmetric pores presumed here, this can increase the residual saturation in the form of singlets to a large degree. Under such a case the universality assumption, for cluster size distribution would fail contrary to the arguments of Larson *et al.* (1981). Here again, due to the constant value of M, altering k does not change the residual saturation.

The bundle of converging-diverging capillaries model is strictly a special case of the present problem with $z \to \infty$. When $z \to \infty$, however,

$$p_c^j = 0 \qquad [61]$$

and

$$R^s(p) = \overline{R}^b(p) = p \quad . \qquad [62]$$

Then [31] and [52] using [17], [59] and [60] simplify to (in the absence of snap-off)

$$S_{w,i} = \int_{\mathscr{H}_i}^{\infty} \mathrm{d}\mathscr{H}_i \int_{\mathscr{H}_i}^{\infty} 24k^2 \mathscr{H}_d \mathscr{H}_i \, \mathrm{e}^{-k\mathscr{H}_d^2} \, \mathrm{e}^{-k\mathscr{H}_i^2} \left(\mathrm{e}^{-k\mathscr{H}_i^2} - \mathrm{e}^{k\mathscr{H}_d^2} \right) \mathrm{d}\mathscr{H}_d \quad . \qquad [63]$$

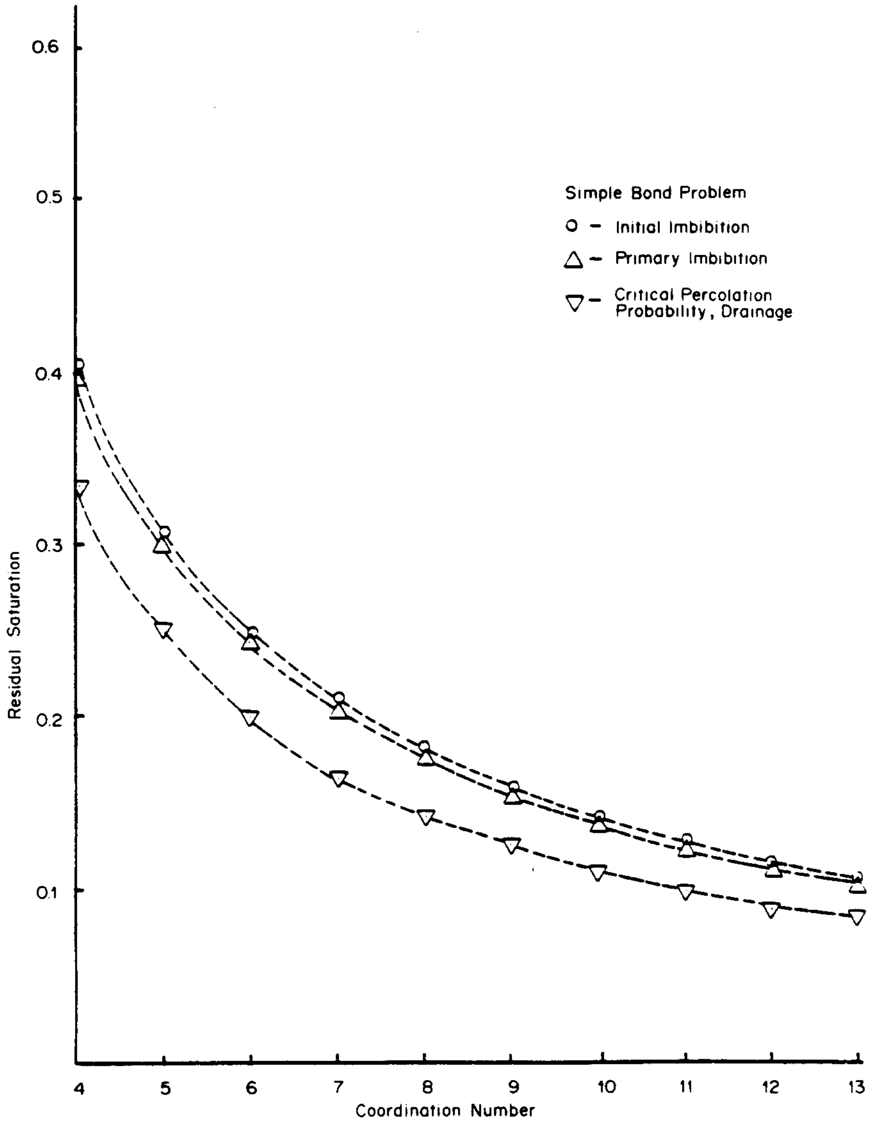


Figure 16. Residual saturations for the simple bond pore.

Integrating [63] we have

$$S_{w,i} = \mathrm{e}^{-3k\mathscr{K}_i^2} \ . \tag{64}$$

Similarly, for the drainage process, the result from [39] is

$$S_{w,d} = 3\,\mathrm{e}^{-k\mathscr{K}_d^2} - 3\,\mathrm{e}^{-2k\mathscr{K}_d^2} + \mathrm{e}^{-3k\mathscr{K}_d^2} \ . \tag{65}$$

The results of the simple bond problem for $z = 10$ and 15 are plotted along with the results of the bundle of capillary tubes model in figure 18. Barring the threshold region which arises due to finite z, these results are extremely close to those of the conventional capillary model of a porous medium. Obviously, due to the absence of the snap-off process, the bundle of tubes model predicts zero residual saturations.

The ratio of p_c^j to the fractional number of pores of the residual phase is a measure of the efficiency of displacement in relation to Bernoulli percolation. It appears that this ratio is fairly constant with respect to z when $z \geq 4$ (figure 19). Extrapolation of this ratio for the bond representation to $z \rightarrow \infty$ suggests that the value approaches 0.76 for primary imbibition and 0.74 for initial imbibition. The corresponding values for the site representation were found to be 0.61 and 0.60. These results are true only when the Bethe lattice percolation

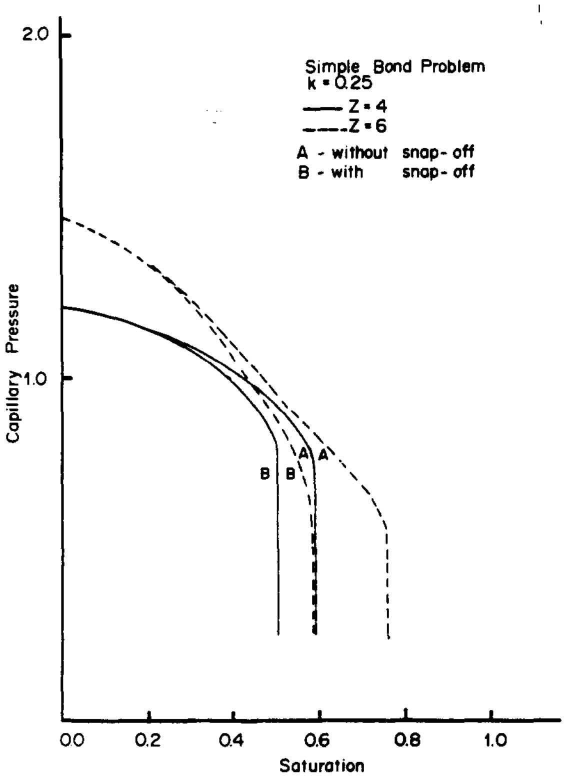


Figure 17. The effect of the snap-off event on the capillary pressure characteristics.

probabilities are used and it appears, from preliminary results, that the values are much closer to one for 3D lattices. When $z = 3$ or 2 (a linear chain), no nonwetting phase is displaced from the medium. For $z = 3$, the critical percolation probability is $\frac{1}{2}$. In this case, when the wetting phase forms an infinite cluster with a saturation equal to 0, the nonwetting phase is totally disconnected, which prevents further removal of it. This is of special relevance to displacement experiments conducted in 2D medium which closely approximate the site representation. Here, if the pore structure as a network forms a planar lattice graph L_2, then use of Fisher's (1961) result that

$$p_c^s(L_2) \geq \frac{1}{2} \qquad [66]$$

implies that no nonwetting phase displacement occurs in a capillary-controlled displacement. Assuming that, for the infinite lattice, the residual saturation results hold for an injection process, the Monte Carlo results of Chandler *et al.* (1982) support this conclusion.

The experimentally observed behavior for both the initial drainage and the primary imbibition processes are shown in figure 20. The data are those of Morrow & Harris (1965) obtained for a bed of glass beads. In spite of the close resemblance between the experiments and the theory (e.g., figure 11) there are some fundamental differences. In the initial drainage curve no threshold behavior is exhibited by the sample. This discrepancy can be explained by the fact that no sample is truly infinite, and that, in any finite lattice graph, the approachability functions do not exhibit sharp critical behavior (Larson & Morrow 1981). The differences when the residual wetting phase saturation is approached are probably due to condition (3) assumed in drainage. In reality, the effect of trapping may be present even when there is an infinite cluster of the wetting phase present in the medium. In addition, the thin film states can theoretically aid in the drainage of the wetting phase and as a result, can mask the predicted sharp critical behavior of the capillary pressure curve.

Although the independent percolation theory approach used here has its obvious advantages, it has severe limitations even in the interpretation of capillary behavior. Noticeable are the approximations that have been used in this study which were essential for

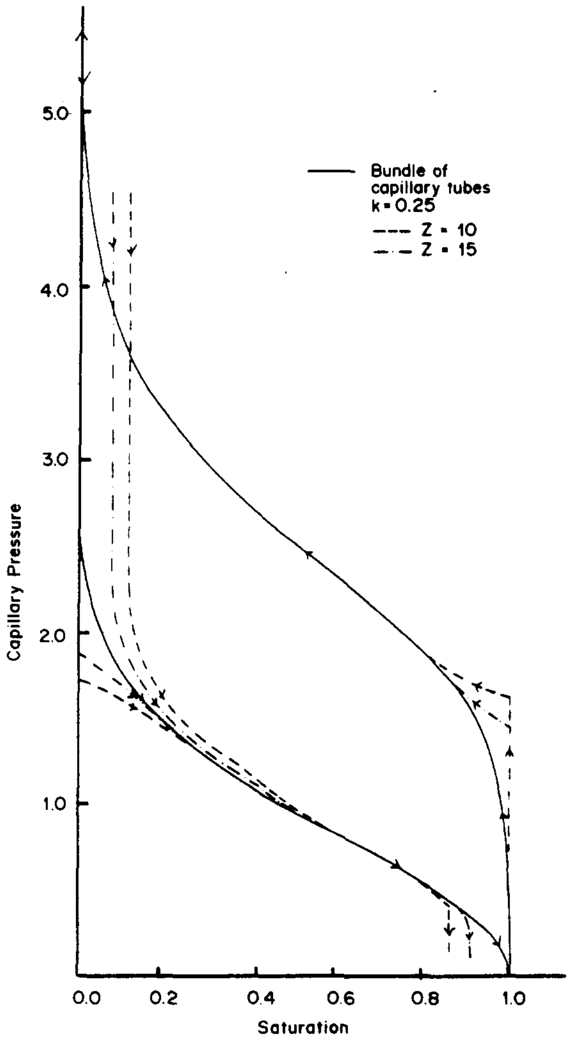


Figure 18. A comparison of the bundle of tubes model with large z results.

the application of the Bernoulli model in understanding two-phase distribution. Clearly this approach cannot take into account the fact that more than two bulk phases can be simultaneously present in a single pore. Also, it cannot consider the change in drainage or imbibition curvatures of a pore with respect to the number of adjoining pores that contain the invading phase. This precludes the interpretation of [8] and [11] from photomicrography alone. Moreover, snap-off events can be far more complicated than visualized here. To model these adequately, Monte Carlo methods provide the best alternative.

SUMMARY

The network representation of a porous medium in terms of an infinite regular lattice graph has been used to study the capillary pressure–saturation relationship. The essential features of the capillary displacement process have been modeled in terms of the Bernoulli percolation model.

The residual number fraction of pores of the nonwetting phase has been shown to be greater than the critical percolation probability of the independent percolation process. Furthermore, the parameters of influence for the residual saturations have been described in terms of the pore shape and the fictitious Bethe lattice coordination number. It is observed that an infinite two-dimensional media under site representation would not show any displacement of the nonwetting phase. The inclusion of the choke-off events results in a substantial increase in the trapping of the nonwetting phase especially for large z lattices, presumably as singlets.

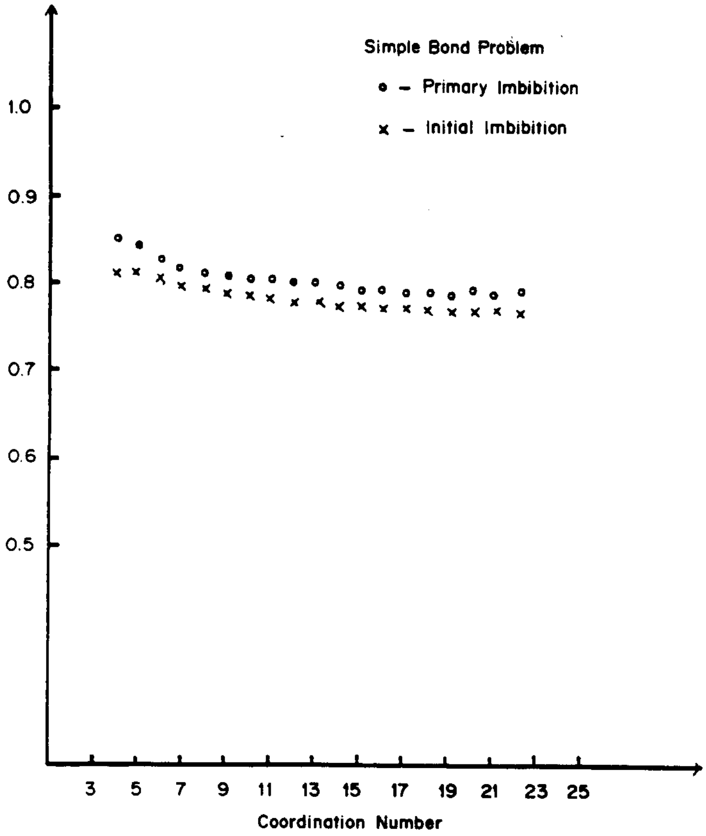


Figure 19. The efficiency of displacement of the nonwetting phase.

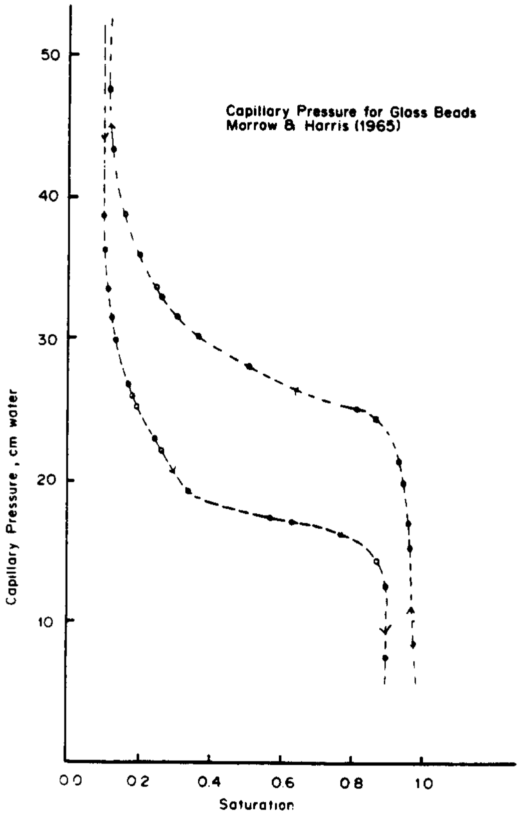


Figure 20. Experimental capillary pressure–saturation relationship.

The theory presented here offers broad guidance in regard to the distinction between displacement and independent percolation. The distinction is not simply the absence of finite clusters of the invading phase. The theory clearly identifies the role trapping plays in this distinction and the effect that isolation of trapped phases has on subsequent efficiency of displacement. Indeed, this makes the treatment developed here approximate, because exact application of percolation ideas becomes impossible. It remains to be seen how the trapping phenomenon affects the conductivity of the two phases and the scaling nature of the trapped clusters.

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APPENDIX A

The mechanical equilibrium at a phase interface dictates (Defay & Prigogine 1966) that

$$\mathscr{P}^*_{c,i} = \gamma\left(\frac{1}{r^*_{1,i}} + \frac{1}{r^*_{2,i}}\right) \qquad [A1]$$

and

$$\mathscr{P}^*_{c,d} = \gamma\left(\frac{1}{r^*_{1,d}} + \frac{1}{r^*_{2,d}}\right) , \qquad [A2]$$

where $\mathscr{P}^*_{c,i}$ and $\mathscr{P}^*_{c,d}$ are the imbibition and drainage capillary pressures in any given pore, γ is the interfacial tension, $r^*_{1,i}$ and $r^*_{2,i}$ are the principal imbibition radii of curvature, and $r^*_{1,d}$ and $r^*_{2,d}$ are the principal drainage radii of curvature for the phase interface. Since the mean curvatures $\mathscr{H}^*_i$ and $\mathscr{H}^*_d$ are given by

$$\mathscr{H}^*_i = \frac{1}{2}\left(\frac{1}{r^*_{1,i}} + \frac{1}{r^*_{2,i}}\right) \qquad [A3]$$

and

$$\mathscr{H}^*_d = \frac{1}{2}\left(\frac{1}{r^*_{1,d}} + \frac{1}{r^*_{2,d}}\right) , \qquad [A4]$$

we have

$$\mathscr{P}^*_{c,i} = 2\gamma\mathscr{H}^*_i \qquad [A5]$$

and

$$\mathscr{P}^*_{c,d} = 2\gamma\mathscr{H}^*_d . \qquad [A6]$$

Defining

$$\mathscr{P}_{c,i} = \mathscr{P}^*_{c,i}/2\gamma\mathscr{H}^*_0 \qquad [A7]$$

and

$$\mathscr{P}_{c,d} = \mathscr{P}^*_{c,d}/2\gamma\mathscr{H}^*_0 , \qquad [A8]$$

with

$$\mathscr{H}_i = \mathscr{H}^*_i/\mathscr{H}^*_0 \qquad [A9]$$

and

$$\mathscr{H}_d = \mathscr{H}^*_d/\mathscr{H}^*_0 , \qquad [A10]$$

we get

$$\mathscr{P}_{c,i} = \mathscr{H}_i \qquad [A11]$$

and

$$\mathscr{P}_{c,d} = \mathscr{H}_d \quad . \qquad [A12]$$

$1/\mathscr{H}^*_0$ is some characteristic length for the pore. It is to be noted that occupation of a pore is permitted when the prevailing capillary pressure $< \mathscr{P}^*_{c,i}$ for imbibition and $> \mathscr{P}^*_{c,d}$ for drainage.

APPENDIX B

We consider the three pore geometries used in the bond representation problem.

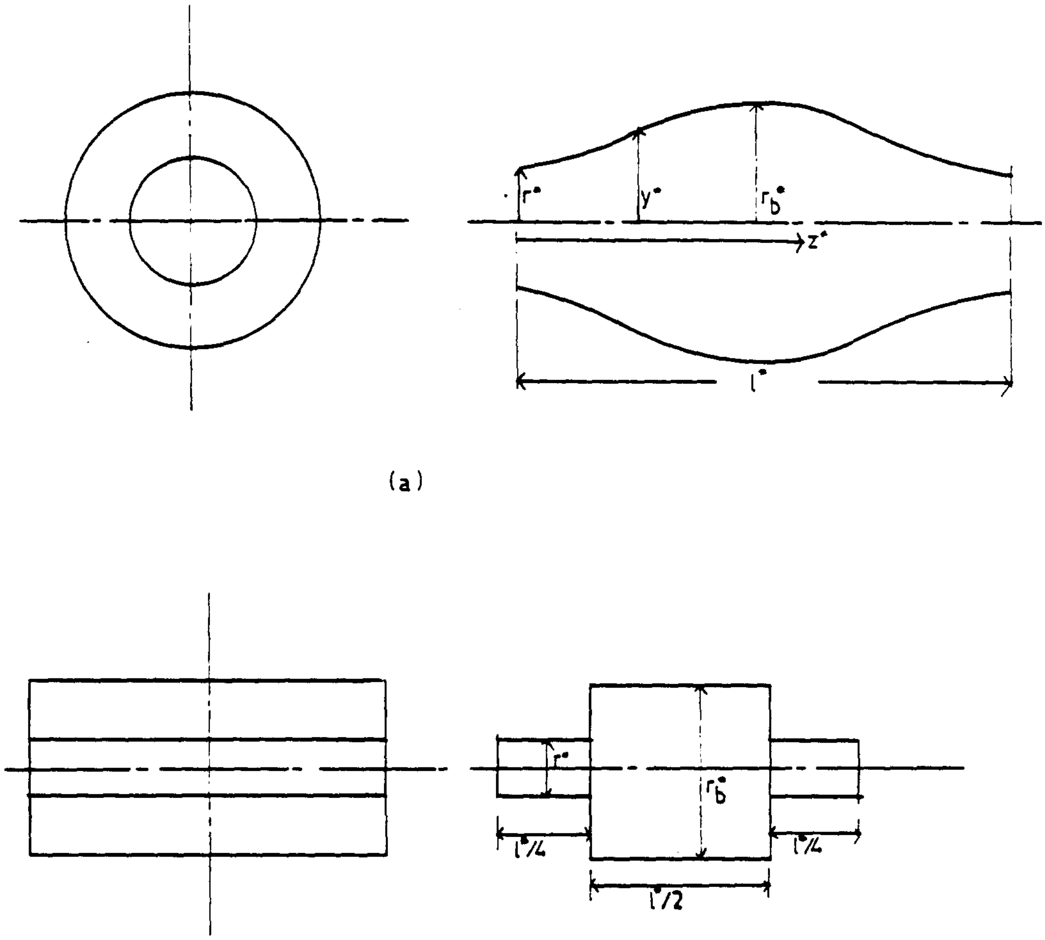
Sinusoidal pore

For this pore shown in figure 21(a), neglecting the effect discontinuous movement has on the imbibition curvatures (Oh & Slattery 1979; Lin & Slattery 1982) we have

$$\mathscr{P}^*_{c,i} = 2\gamma/r^*_b = 2\gamma\mathscr{H}^*_i \quad , \qquad [B1]$$

with $\mathscr{H}^*_i = 1/r^*_b$ and

$$\mathscr{P}^*_{c,d} = 2\gamma/r^* = 2\gamma\mathscr{H}^*_d \quad , \qquad [B2]$$



(a)

(b)

Figure 21. (a) The sinusoidal pore; (b) the rectangular pore.

with $\mathscr{H}_d^* = 1/r^*$. The volume of the pore V^* then is

$$V^* = \pi \int_0^{l^*} y^{*2}\, dz^* \quad , \qquad [B3]$$

with

$$y^* = r^* + (r_b^* - r^*) \sin (\pi z^*/l^*) \quad . \qquad [B4]$$

From [B3]

$$V^* = \frac{\pi l^*}{\mathscr{H}_d^{*2}}\left(\frac{3}{2} - \frac{\pi}{4}\right) + \frac{\pi l^*}{2\mathscr{H}_i^{*2}} + \frac{l^*}{\mathscr{H}_d^* \mathscr{H}_i^*}(4 - \pi) \quad . \qquad [B5]$$

Fixing the length l^* of the pores and choosing $\mathscr{H}_0^* = 1/l^*$ and $V = V^* \mathscr{H}_0^{*3}$, the volume assigned to the sinusoidal pore is

$$V(\mathscr{H}_d, \mathscr{H}_i) = \frac{\pi}{\mathscr{H}_d^2}\left(\frac{3}{2} - \frac{4}{\pi}\right) + \frac{\pi}{2\mathscr{H}_i^2} + \frac{(4 - \pi)}{\mathscr{H}_d - \mathscr{H}_i} . \qquad [B6]$$

Rectangular pore

For this pore [figure 21(b)] when $l^* >> r^*, r_b^*$ the equations of capillary pressure (Lenormand *et al.* 1983) reduce to those of parallel plates (Carman 1941). Then,

$$\mathscr{P}_{c,i}^* = 2\gamma/r_b^* = 2\gamma \mathscr{H}_i^* \qquad [B7]$$

and

$$\mathscr{P}_{c,d}^* = 2\gamma/r^* = 2\gamma \mathscr{H}_d^* \quad , \qquad [B8]$$

the volume of the pore being

$$V^* = \frac{l^{*2}}{2}\left(\frac{1}{\mathscr{H}_d^*} + \frac{1}{\mathscr{H}_i^*}\right) \quad . \qquad [B9]$$

Defining $\mathscr{H}_0^* = l^*$, $V = V^*/l^{*3}$, we have the result that

$$V(\mathscr{H}_d, \mathscr{H}_i) = \frac{1}{2}\left(\frac{1}{\mathscr{H}_d} + \frac{1}{\mathscr{H}_i}\right) \quad . \qquad [B10]$$

Simple Bond Problem

This is a trivial case where $\mathscr{P}_{c,d} = \mathscr{H}_d$, $\mathscr{P}_{c,i} = \mathscr{H}_i$ and $V(\mathscr{H}_d, \mathscr{H}_i) = 1$.